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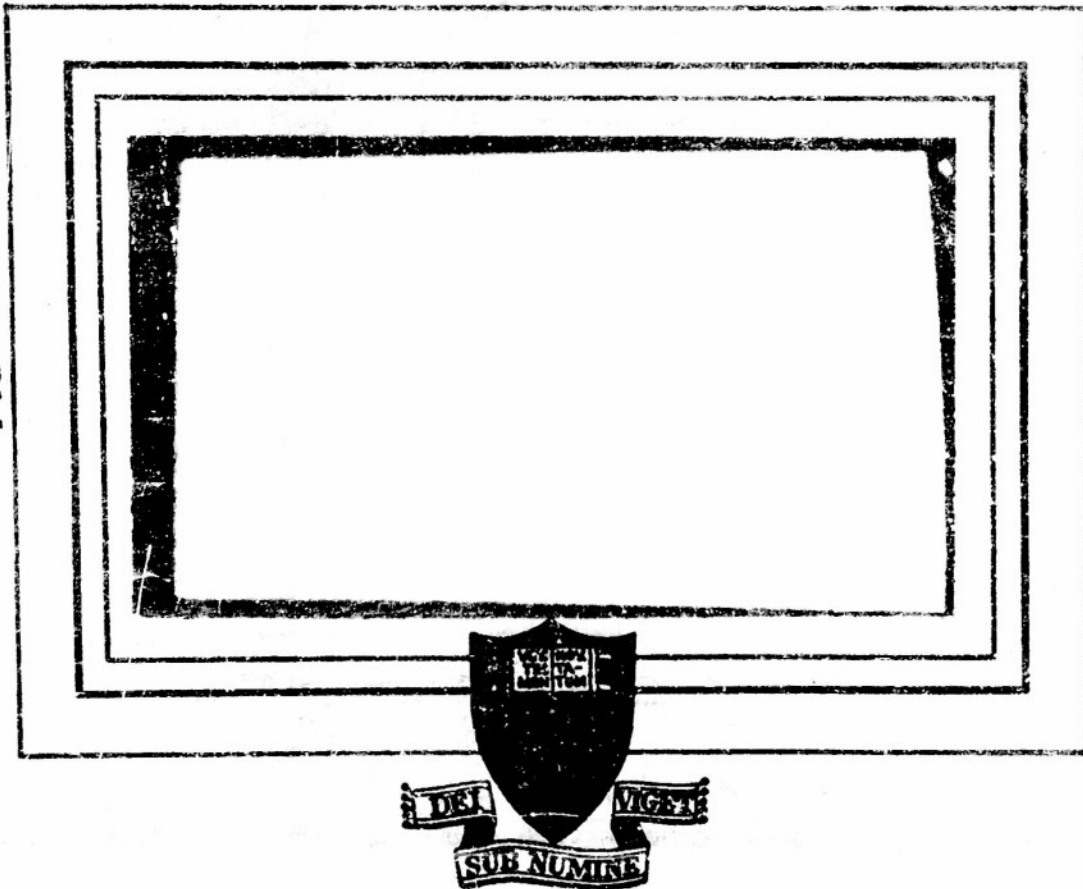
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THE TRANSVERSE CURVATURE EFFECT IN
COMPRESSIBLE AXIALLY-SYMMETRIC
LAMINAR BOUNDARY LAYER FLOW

by

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TABLE OF CONTENTS

	Page
ACKNOWLEDGEMENT	i
SUMMARY	ii
LIST OF SYMBOLS	iv
1. INTRODUCTION	1
1.1 Preliminary Considerations and Transverse Curvature Parameter	1
1.2 Description and Range of Main Flow Regions	4
1.3 Review of Previous Work for Δ/r_0 Less Than But Not Necessarily Small Compared to Unity	12
1.4 Plan of the Present Investigation	16
2. THEORY	18
2.1 Boundary Layer Equations for Axial Symmetry and Particular Integrals of the Energy Equation for $Pr = 1$	18
2.2 Reduction of Equations to Almost Two-Dimensional Form and Physical Interpretation of Transverse Curvature Terms	22
2.3 The Incompressible Plane and Similarity Considerations	29
2.4 Summary of Mathematical Development	38
3. SOLUTION OF EQUATIONS	39
3.1 Asymptotic Expansions, Boundary Conditions, and Zero th Order Equations	39
3.2 First Order Equations and Admissable Body Classes	41
3.3 Some Numerical Results for the Cone and Cylinder for $Pr = 1$	45
4. FUTURE INVESTIGATIONS	49
CONCLUSIONS	51
REFERENCES	55

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SUMMARY

The viscous transverse curvature effect in compressible axially-symmetric laminar boundary layer flow has been investigated, and it is found that the effect is characterized by the parameter Δ/r_0 which is essentially the ratio of the boundary layer thickness to body radius. It is shown that the Busemann and Crocco integrals of the two-dimensional energy equation for $Pr = 1$, are still valid for axially-symmetric flow in which the transverse curvature effects are considered. By a generalization of Mangler's transformation it is then shown that the boundary layer equations are reducible to an almost two-dimensional form, making the analysis simpler for two asymptotic flow regions characterized by $\Delta/r_0 \gg 1$ and Δ/r_0 less than or of the order of unity. It is with the latter region that the present paper is primarily concerned, and for this case it is shown that the additional term in the momentum and energy equation, which differentiates it from the two-dimensional form, behaves like an axial pressure gradient. On this basis the results of previous authors are interpreted. Except for the case of a "near paraboloid" with zero pressure gradient where "similar" profiles can be found for all values of Δ/r_0 it is necessary to obtain the "exact" solutions in the range where Δ/r_0 is less than or possibly of the order of unity by means of asymptotic expansions in ascending powers of a parameter which is small compared to unity but proportional to Δ/r_0 . It is shown how the asymptotic solutions for the velocity and temperature can be found for "zero pressure gradient" when

the body shapes go like $r_0 = ax^n$ and $r_0 = ae^{bx}$. The zeroth approximation is the Mangler result.

The first order correction to the Mangler formulation for $Pr = 1$ shows, that at least in the case of the cone and cylinder, the effect on both the skin friction coefficient and heat transfer rate can become appreciable in the range where Δ/r_0 is less than or of the order of unity. At a constant Δ/r_0 , the effects are increased in magnitude when either the ratio of wall to free stream temperature, or Mach number, is increased. Also, all other conditions being equal for the same value of Δ/r_0 the skin friction coefficient and heat transfer increase on the cylinder is greater than that on the cone.

LIST OF SYMBOLS

The subscript "e" denotes quantities in the inviscid external flow, and the subscript " ∞ " denotes values in the undisturbed free stream far from the body. The subscript "w" refers to values of the physical quantities at the wall and the subscript "M" refers to the value given by the Mangler formulation.

- x coordinate measured along the body surface with origin at the nose
- y coordinate normal to surface
- Θ azimuthal angle
- α angle tangent to meridian profile makes with body axis
- r_0 distance of any point on the body $(x, 0, \Theta)$ to the central axis.
 $r_0 = r_0(x)$
- r distance from any point (x, y, Θ) to the axis of symmetry, in the present paper $r(x, y) = r_0(x) + y \cos \alpha$
- K_1 longitudinal curvature in meridian plane
- K_2 transverse curvature in plane perpendicular to flow, $1/r$ for axial symmetry
- u component of velocity in x direction
- v component of velocity in y direction
- p static pressure
- ρ mass density
- T absolute temperature
- L characteristic reference length of body
- δ boundary layer thickness

δ^* boundary layer displacement thickness, $\int_0^{\infty} \left[1 - (\rho u / \rho_e u_e) \right] dy$

Δ $\delta^* \cos \alpha$

R gas constant per gram

c_p specific heat at constant pressure

c_v specific heat at constant volume

γ ratio of specific heats, c_p/c_v

μ coefficient of viscosity

ν coefficient of kinematic viscosity

k coefficient of thermal conductivity of gas

h enthalpy, $c_p T$

M_e Mach number, $u_e / \sqrt{\gamma R T_e}$

$(Re_x)_e$ Reynolds number based on the length x , and quantities in the inviscid external flow, $u_e x / \nu_e$

Pr Prandtl number of the gas, $c_p \mu / k$

C_e factor of proportionality in the equation $(\mu/\mu_e) = C_e (T/T_e)$

$$\bar{x} \quad \int \frac{\bar{x}_0^2}{L^2} dx$$

$$\bar{y} \quad \int \frac{r(x,y)}{L} dy = \frac{r_0 y}{L} + \frac{y^2 \cos \alpha}{2L}$$

$$\bar{v} \quad \frac{rL}{r_0^2} v + \frac{L^2}{r_0^2} \frac{\partial \bar{y}}{\partial x} u$$

$$\bar{y} \quad \int \frac{p}{\rho_e} d\bar{y}$$

$$\eta \quad (u_e / C_e \nu_e \bar{x})^{1/2} \bar{y}$$

$$\xi \quad (C_e \nu_e / u_e)^{1/2} \frac{L \cos \alpha}{r_0^2} \sqrt{\bar{x}}$$

f defined by $\frac{u}{u_e} = \frac{\partial f}{\partial \eta}$

λ T/T_e

τ shear stress, μu_y

c_f local skin friction coefficient, $\tau_w / \frac{1}{2} \rho_e u_e^2$

q local rate of heat transfer from the surface per unit area per unit time, $-k_w (T_y)_{y=0}$

$$\mathcal{R} \left\{ \frac{2}{\tau_0} + \frac{\tau_0''}{(1 - \tau_0'^2)} \right\} \frac{\tau_0'}{\tau_0^2} \int \tau_0^2 dx$$

1. INTRODUCTION

1.1 Preliminary Considerations and Transverse Curvature Parameter

The steady laminar boundary layer on an unyawed body of revolution differs from that on a two-dimensional shape in that the axially-symmetric boundary layer must not only grow in thickness with distance along the surface, but in addition must also spread circumferentially as it grows. Clearly therefore, the rate at which the body circumference changes with length will be the axially-symmetric geometrical factor which will determine the characteristics of the retarded viscous layer. The body geometry effect can therefore be considered to manifest itself through the two surface curvatures shown in Fig. 1 for a pointed body of revolution. The first is the longitudinal curvature in a meridian plane, denoted by K_1 , while the second is the transverse curvature, K_2 , of the body in a plane perpendicular to the flow. Now, it is evident that the longitudinal surface curvature is a quantity which is associated not only with axial symmetry, but also with any curved surface in two-dimensional flow. In the usual treatment of boundary layer problems, it is generally assumed that δK_1 and $\delta^2 \frac{dK_1}{dx}$ are small compared to unity, (here δ is the boundary layer thickness), in which case the effects of longitudinal surface curvature are negligible. Of course, these conditions impose certain restrictions on the body shape, and the effect of removing them has been studied by Murphy¹

for incompressible flow with zero pressure gradient. However, so far as the present paper is concerned, this longitudinal curvature effect will be neglected.

The transverse curvature of the body, K_2 , in a plane perpendicular to the flow, is distinguished by the fact that it arises only from the three-dimensional nature of the problem. Of course, for a body of revolution the section in any transverse plane is by definition a circle, so that here the curvature is simply the inverse of the body radius, r_0 , at any position along the axis. It is the purpose of this work to examine what effect the introduction of this transverse curvature has on the values of such quantities as the viscous shear and heat flux particularly at the body surface. The present paper indicates the fundamental physical and mathematical ideas involved; the more detailed computations and numerical results being reserved for a forthcoming report. In addition, the future work will contain a more complete analysis of the effect of pressure gradient in axially-symmetric flow and its "interaction" with the transverse curvature effect. It should be noted at this point that the hypersonic self-induced pressure effect^{2, 3, 4} generated by the interaction of the longitudinal curvature of the viscous layer with the external flow, could under certain circumstances become more important than the transverse curvature effect itself. Although this important question certainly requires further investigation, it is beyond the scope of the present paper except for certain brief comments.

Cheng⁵ in unpublished lecture notes has shown that in general a solution to the boundary layer problem, in which the transverse curvature

effect is considered, is not possible by a single transformation of the equations to a two-dimensional form without some assumption as to the magnitude of the curvature effect. Clearly, a measure of this magnitude so far as the viscous flow is concerned would be given by the ratio of the transverse geometrical body curvature to the transverse curvature of the viscous layer itself. Because of the axial symmetry the rate of change of curvature with respect to the azimuthal angle is zero, so that while two parameters measure the longitudinal curvature effect, only one is needed to characterize the transverse curvature effect.

At this point let us define the coordinate system (x, y, Θ) where the body surface is given by $y = \text{constant} = 0$, x is the distance measured along the body from some reference point which in the present paper is taken to be the nose of the body ($x = 0$), and Θ is the azimuthal angle, (see Fig. 1). Let $r = r(x, y)$ be the distance from any point (x, y, Θ) to the axis of symmetry, and suppose $r_0 = r_0(x)$ is the distance of any point on the body $(x, 0, 0)$ to the central axis. If as assumed, the thickness of the boundary layer is small compared with the longitudinal radius of curvature, $1/K_1$, then at any point in the boundary layer

$$r(x, y) = r_0(x) + y \cos \alpha$$

where α is the angle the tangent to the meridian profile makes with the axis. Although the characteristic viscous length is the boundary layer thickness δ , it is more convenient to deal with the displacement

thickness, δ^* , because it is capable of precise definition.* It follows that the curvature ratio is given by

$$\frac{K_2}{K_2}_{\text{visc}} = \frac{r_{\delta^*}}{r_o} = 1 + \frac{\delta^* \cos \alpha}{r_o} = 1 + \frac{\Delta}{r_o}$$

Here, $\Delta = \delta^* \cos \alpha$ is the projection of the displacement thickness onto the transverse plane, (see Fig. 2). Thus, so far as the present problem is concerned, the significant parameter is the ratio of the boundary layer thickness, or more precisely the displacement thickness, to the body radius.** Therefore, certain simplifications in the complete axially-symmetric boundary layer equations of continuity, momentum, and energy should be possible, depending upon the value of Δ/r_o in comparison to unity.

1.2 Description and Range of Main Flow Regions

Referring to Fig. 2 - in which a cone in supersonic flow is chosen to illustrate the body of revolution - two main asymptotic flow regions over a given body can be distinguished primarily upon the basis of the

* See discussions p. 123, Ref. 6, p. 18 ff., Ref. 2, and p. 381, Ref. 7. By definition the displacement thickness in compressible flow is given by

$$\delta^* = \int_0^{\infty} \left[1 - \left(\rho u / \rho_e u_e \right) \right] dy$$

** This statement assumes $\cos \alpha$ to be unity so that $\Delta \approx \delta^*$, a condition which is almost true for the slender bodies which are under consideration, except near a blunt forward stagnation point.

order of Δ/r_0 .^{*} For the body of Fig. 2 the two main flow regions are respectively: I, a "nose" region distinguished by the fact that $\Delta/r_0 \gg 1$, and II, a "downstream" region where Δ/r_0 is "of the order or less than unity". The application of the term asymptotic flow region to the range where Δ/r_0 is of the order or less than unity might seem somewhat unjustified, since by accepted definition one would require that Δ/r_0 , or at least $(\Delta/r_0)^2$ be small compared to unity. However, the region is referred to as asymptotic because, as will be shown, the asymptotic expansion parameter for the physical variables, which is directly proportional to Δ/r_0 , is in general small compared to unity when Δ/r_0 is of the order or less than unity.

The two regions I, and II are separated by a "transition" zone where Δ/r_0 is intermediate between the nose and downstream region. In the present investigation the nose of the body is considered mathematically sharp and the "immediate" nose region where slip, temperature jump and other kinetic effects could become important is neglected. This would exclude from consideration a zone whose extent for compressible flow in terms of the local Reynolds

* Here, a pointed nose body of revolution is considered. In the case of a cylinder with its generators parallel to the flow, (sufficiently slender so that the Δ/r_0 effect enters) the regions will be reversed in their relation to distance from the nose when compared to the pointed body.

number would be of the order of $Re_x \sim 10 M^2$ where here M is the local Mach number. The overall flow problem will then be treated in such a way that all possible information about the nature of the flow regions downstream of the "immediate" nose region is obtained, which does not depend on the detailed history of the flow in this region. However, as is usual in boundary layer theory the origin of the coordinate system will be taken at $x = 0$, or the nose where the boundary layer thickness is supposed to be zero.

Before discussing the regions in more detail, a criterion is needed to determine the extent of the various zones. This can be obtained from the following relation for an insulated cone of half-angle α , in supersonic flow^{*}

$$(1) \quad \left(\frac{\Delta}{r_0} \right)_M = \frac{(0.994 + 0.275 M_e^2) \sqrt{C_e}}{\tan \alpha \sqrt{(Re_x)_e}}$$

Here, the specific heat ratio has been taken equal to 1.4, the subscript "e"

* This relation was obtained using the flat plate, compressible laminar boundary layer solution for constant wall temperature and specific heat, a linear viscosity-temperature relation, and $Pr = 1$, (see e.g., Ref. 2).

$$(1a) \quad \int \frac{(Re_x)_e}{C_e} \cdot \frac{\delta^*}{x} = 1.721 \frac{T_{wall}}{T_e} + 0.332 (\gamma - 1) M_e^2$$

From Mangler's transformation⁸, division by $\sqrt{3}$ gives the axially-symmetric value for a cone in supersonic flow, valid in the range where $\Delta/r_0 < 1$. This solution is examined when the zone II is considered, its use in this instance is only to permit an estimate of the lateral extent of the regions being studied.

refers to the inviscid values of the physical quantities downstream of the nose shock, and C_e is a parameter which appears in the linear viscosity-temperature law to be introduced later. It must be emphasized that the above relation is only intended to give the orders of magnitude involved. Thus, it can be seen that the more slender the cone, the closer to the nose, or the higher the flight speed, the greater will be the transverse curvature effect in compressible flow.

Figure 3 is a plot of Eq. (1), with Δ/r_0 as the parameter, where the abscissa is the Mach number behind the conical shock, and the ordinate is $\frac{1}{\tan \alpha} \sqrt{C_e / (Re_x)_c}$. It is important at this point to note that it is the ordinate parameter when divided by $\sqrt{3}$ which turns out later to be the proper asymptotic expansion parameter, (ξ) , for the cone. In this figure, $(Re_x)_c$ is the Reynolds number based on conditions downstream of the shock and distance along the cone surface. The Reynolds number is to be interpreted as denoting the lateral extent of the region in which the transverse curvature effect is of importance for the particular Δ/r_0 range being studied. Strictly speaking of course, the cone solution should only be valid for supersonic flow with an attached shock, in order to satisfy the condition that the pressure gradient be zero, however the values are carried down to zero Mach number for continuity purposes.

Figure 3 is even more general if it is recognized that to the same order of approximation it could be used to discuss the cylinder

by the artifice of dividing the ordinate scale by $\sqrt{3}$, and considering $\tan \alpha$ to be replaced by r_0/x where r_0 is the cylinder radius and x the distance from the nose. In this case, the subscript "e" would denote the conditions in the undisturbed free stream far from the body. With these substitutions the ordinate is then exactly the asymptotic expansion parameter, (ξ) , which will be used later for the cylinder.

What is important to note is that for $\Delta/r_0 \approx 1$, the asymptotic expansion parameter ξ , (for both the cone and cylinder), will have a maximum value of around $1/4$ in supersonic flow. Furthermore, its value will drop off rapidly with either increasing Mach number or decreasing Δ/r_0 .

The nose region (1) is characterized by the fact that $\Delta/r_0 \gg 1$. In other words, in this zone the stress term arising from the transverse curvature becomes of the same order of magnitude as the usual viscous stress term in the momentum equation. An analogous statement holds true for the heat flux terms in the energy equation. Obviously the above conditions are met for only certain ranges of Mach and Reynolds numbers. In actuality $\Delta/r_0 = 10$ is most probably the lower limit for what could be taken to be a value of the transverse curvature parameter which is supposed to be large compared to unity. Therefore, if the nose region is considered to be represented by that area above the line $\Delta/r_0 = 10$, then it follows almost immediately that the actual practical range in which this region has any importance may be rather limited below a Mach number of 10. In fact if $\Delta/r_0 = 10$, the cone half-angle has to be as small as 1° even though the Mach number be as high as 7.7 in order that the effect extend to a Reynolds

number of 10,000, ($C_e \approx 1$, $\tan \alpha \sqrt{(Re_x)_e} = 1.75$). However for Mach numbers greater than about 4, when the self-induced hypersonic viscous parameter $\bar{\chi} = \sqrt{C_e} M_e^3 / \sqrt{(Re_x)_e} = 1$,^{2,3,4} the hypersonic viscous effects can become important. Using $\bar{\chi} = 1$ as a criterion, calculations and Fig. 3 show the nose region to be affected by the self-induced pressure gradient even for cones as slender as 1° . It is likely therefore, that above $M_e \approx 4$ the hypersonic viscous phenomena may be at least as important as the transverse curvature effects for the region in which $\Delta/r_0 \gg 1$. In any event a solution for the nose region is useful in the sense that it serves as the limiting case with which to bridge the "transition" zone. It could also conceivably serve as an aid in checking any approximate solution intended to cover the spectrum of Δ/r_0 .

Recently this regime has been considered by Stewartson⁹ who examined the case of the "infinitely" thin cylinder in the absence of pressure gradients in incompressible flow. He found that for this limiting case the leading term in the asymptotic expansion of the axial velocity component is simply the main stream velocity and the next term is the same as that derived by Oseen's method.¹⁰ Mark working under the direction of Lees has obtained "exact" solutions to the large Δ/r_0 problem for the incompressible flow over a paraboloid of revolution with zero pressure gradient, utilizing Stewartson's result that the limiting form of the flow is of the Oseen type. Knowing the exact solution it is understood that he has developed an approximate momentum integral method to solve the large Δ/r_0 problem by using a

Kármán-Pohlhausen technique which is modified by adding an appropriate logarithmic term to the velocity distribution. Cooper and Tulin¹⁵ however, say that Pretsch¹¹ found similar^{*} velocity profiles in treating the same problem as that considered by Stewartson. This result is in direct contradiction to the one found in Ref. 9. Unfortunately at the time of writing the authors have been unable to obtain Pretsch's paper in an effort to resolve the apparent contradiction. So far as the present paper is concerned this region will not be considered further except insofar as the boundary layer equations which are valid for it are written down, and also insofar as certain general solutions are obtained which hold for all values of Δ/r_0 . For example, reference is made here to particular integrals of the energy equation and to the "zero pressure gradient" flow over a nearly parabolic body of revolution for which the equations are noted.

Just downstream of the nose region is the transition zone in which Δ/r_0 is neither large compared with one, nor of the order of unity. In Fig. 3, this zone can be considered to be encompassed by the area contained between the curves of $\Delta/r_0 = 1$ and 10. Here, in general, analytic solutions of the equations of motion are difficult to obtain, and it is probable that an extension of an approximate momentum-integral technique to this domain would be required in order to determine a solution. Of course if the large Δ/r_0 result is known, it may also be possible to "link up" a solution through this region.

In region II Δ/r_0 is of the order or less than unity. This zone is

* Similar in the sense that the velocity distribution function is a function of a single variable of the form y times a function x where x and y are respectively the distances measured along and perpendicular to the boundary.

characterized by the fact that the effects produced by the transverse curvature can be considered to be essentially a perturbation of a flow which, in the limit of Δ/r_0 very much less than unity, approaches a two-dimensional pattern. That is, it is clear that as one proceeds downstream along the cone the transverse curvature effects must decay. In Fig. 3 this downstream region may be considered as the area lying below the curve of $\Delta/r_0 = 1$. It is quickly evident from the curves that the "weaker" transverse curvature effects can extend over the major portion of the body, (say a cone whose half-angle is as much as 5°), for Mach numbers which are not large. Actually this domain can be thought of as being divided into three sub-regions each one of which is simply a limiting case of the other.

In the sub-region (3) (Fig. 2) very far downstream of the nose, $r_0 \rightarrow \infty$ so that Δ/r_0 tends to zero, and $K_2 / K_2 \xrightarrow{\text{body visc}} 1$.

Here, the effect of the axial-symmetry is negligible and the flow approaches a two-dimensional pattern. In practice however, it is unlikely that this domain would be reached before transition to turbulent flow took place. For example, on a 5° cone at a Mach number of 3, the Reynolds number is already 1.5×10^7 for $\Delta/r_0 = 0.01$.

Somewhat further upstream, in the sub-region (2), although Δ/r_0 is small compared to unity, (say 0.1 or less), the flow can nevertheless no longer be considered two-dimensional in character. By our definition then, this would be the area lying below the curve of $\Delta/r_0 = 0.1$ in Fig. 3. Here, the circumferential spreading of

the viscous layer must be considered. However the approximation can be made in the boundary layer equations that $r(x, y) \approx r_0(x)$, in which case the momentum and energy equations become two-dimensional in form and the continuity equation is considerably simplified. Mangler⁸ has shown that in this case a direct transformation of the compressible boundary layer equations for axially-symmetric flow to those of a two-dimensional flow is possible when the gas is perfect with constant specific heats. Actually although this transformation was not expressed formally until 1945, it had been used in essence as far back as 1908 by Boltze¹² in the treatment of axially-symmetric laminar flows (for other examples see Goldstein⁶).

Proceeding still further upstream the sub-region ① is reached, where the boundary layer thickness begins to approach or become of the order of the body radius, (in the spirit of the foregoing definition of order of magnitude), in which case the transverse curvature takes on still more importance. Here, we would be concerned with that area in Fig. 3 which lies between the curves $\Delta/r_0 = 0.1$ and $\Delta/r_0 = 1$. One must however continue to bear in mind that any treatment valid for this region must actually encompass those zones downstream of it. The present paper then is primarily concerned with the investigation of the compressible laminar boundary layer flow in such a region, where Δ/r_0 is less than or possibly of the order of unity.

1.3 Review of Previous Work for Δ/r_0 Less Than But Not Necessarily Small Compared to Unity

As far as the present authors know, all the so-called exact approaches to the above problem have been restricted to incompressible cylindrical

flow. In this case, the body radius r_0 is a constant, and $r = r_0 + y$, (see Fig. 1). The first solution was given by Atkinson and Goldstein (Ref. 6, p. 304) in their investigation of the internal steady incompressible laminar boundary layer near the entry of a cylindrical pipe. Their analysis also included the effect of a self-induced pressure gradient due to the growth of the viscous layer in the pipe. A solution was obtained by expanding the stream function in an asymptotic series in ascending powers of \sqrt{x}/r_0 , where the coefficients in the series were only functions of the variable $\left(\frac{r^2 - r_0^2}{r_0}\right) / \sqrt{x}$.

In this manner the problem was reduced to the solution of a series of ordinary differential equations, the first of which was the non-linear Blasius equation, while the remaining ones were third order linear, inhomogeneous equations. It should be noted at this point, that this technique employed the idea of expanding the physical quantities in powers of the transverse curvature parameter $\Delta/r_0 \sim \sqrt{x}/r_0$. Whether or not this was implicitly recognized by the authors is of course not known. In any event, it is clear that the first term represents a flow which is two-dimensional in character, while the succeeding terms in the series characterize the effect of the transverse curvature as well as the self-induced pressure gradient.

13

In 1951, Seban and Bond simply extended the preceding analysis to include the "incompressible" energy equation. In their treatment the problem considered was that of the exterior steady incompressible flow over a cylinder with constant pressure, that is, the self-induced

pressure gradient was taken to be zero. The variables used were the same as those introduced by Atkinson and Goldstein. Their calculations showed that the transverse curvature effect increased the local skin friction coefficient and heat transfer rate, and that this increase could become appreciable. However, the displacement thickness of the boundary layer is only slightly reduced in comparison with that of the flat plate. Furthermore, the recovery factor appears to remain unaffected within the probable numerical errors. The present authors will have more to say regarding the significance of these results at a later point in the paper.

Sowerby and Cooke¹⁴ performed the same analysis of the momentum equation for the steady incompressible flow over a cylinder as did Seban and Bond. In addition, they also treated the analogous non-steady "Rayleigh problem" of the infinite circular cylinder started impulsively from rest in its own plane. They found that there is a regime in which the boundary layer growth is independent of the space variable x , but dependent on the time variable t , so that here x is replaced by t and $\Delta/r_0 \sim \sqrt{t}/r_0$. As one would expect, the solution of this linear problem turns out to be an asymptotic expansion in powers of \sqrt{t}/r_0 for the physical quantities, such as the viscous shear.

Finally, Cooper and Tulin¹⁵ linearized the incompressible boundary layer equations of motion in the cylindrical polar coordinate system, and again determined the solution to the problem of the steady flow over a cylinder with no pressure gradient. In this case a general functional solution which essentially is valid for all values of Δ/r_0 could be given

in terms of Bessel functions of zero order. However, only the asymptotic solution for the Δ/r_0 range considered in the present paper was given. The fact does not seem to have been noted that their asymptotic expansion was in powers of a quantity proportional to the physical transverse curvature parameter Δ/r_0 . They also found a general solution to the linearized problem of the unsteady motion without a pressure gradient, of a cylinder in axial flow which starts from rest at time $t = 0$. Here, only the particular case of the impulsive start was treated in detail, with analogous results, as pointed out previously, to the steady flow problem. A general solution for a large class of pressure gradients on a cylinder was also obtained, although only the Falkner-Skan¹⁶ type was treated in detail. Again, by means of an asymptotic expansion in a quantity proportional to Δ/r_0 they showed that the transverse curvature effect increased the wall shear stress for both favorable and adverse pressure gradients. In the range of $\Delta/r_0 \leq 1$, $|\xi| \ll 1$, they found that the increase in wall shear on the cylinder for favorable gradients when compared with the shear on the flat plate for the same favorable gradient, is less than the increase on the cylinder in uniform flow when compared with the flat plate in uniform flow. The converse is true for adverse gradients. The significance of these results will be discussed later in the paper. For the case of a constant external velocity gradient over the cylinder they showed that similar profiles existed, and derived the ordinary differential equation defining them.

1.4 Plan of the Present Investigation

In the present paper a theory is presented of the transverse curvature effect on compressible laminar boundary layer flow over bodies of revolution. First, it is shown that whenever the Prandtl number is equal to unity, in spite of the transverse curvature effect the Busemann integral of the energy equation¹⁷ is still valid for an insulated surface with an isoenergetic free stream regardless of the pressure gradient. Furthermore, it is also shown that the Crocco integral of the energy equation¹⁸ holds for the constant surface temperature zero pressure gradient case.

Next, for constant specific heat and a perfect gas a generalization of Mangler's transformation is considered, which also alters the x, y scale to some new scale \bar{x}, \bar{y} in a manner which depends upon the body shape. This transformation reduces the boundary layer equations to an almost two-dimensional form. The additional term which arises in both the momentum and energy equation is shown to have the same effect as an external favorable pressure gradient. On this basis, results obtained by previous authors are interpreted and discussed, while the results that might be expected from the present work are also reviewed.

The assumption of a linear viscosity-temperature relationship is then made, so that by means of Howarth's transformation¹⁹ which is an alteration of the scale in the direction normal to the body surface, the equations are reduced to an "incompressible form". For simplicity the problem is now restricted to the case of zero pressure gradient, although the methods developed are applicable to a more general class of flows. The validity of

restricting the problem to constant pressure flows is briefly discussed in spite of the fact that the main burden of this study is left for a forthcoming report. Finally, the partial differential equations are transformed to an \bar{x} , $Y/\sqrt{\bar{x}}$ coordinate system, where here Y is the stretched normal coordinate. From these equations it follows directly for a particular class of bodies not too different from a paraboloid of revolution, that the momentum and energy equations reduce to ordinary differential equations in the single variable $\eta \sim Y/\sqrt{\bar{x}}$. It is the equation obtained in this case for incompressible flow which is being treated by Mark¹⁰.

For more general shapes solutions can be obtained by expanding all physical quantities in asymptotic series in powers of a quantity proportional to Δ/r_0 , where the coefficients are functions only of the variable η . Thus any physical quantity $G(x, y)$ can be written as

$$(2) \quad G(x, y) = G_0(\eta) + \xi G_1(\eta) + \xi^2 G_2(\eta) + \dots$$

where $\xi \sim \Delta/r_0 \ll 1$, and where G_0 , G_1 , G_2 , etc., are the zeroth, first, second, etc., order approximations respectively. In particular the velocity and temperature distribution functions are expanded in this way, and the partial differential equations characterizing the motion can be reduced to a double infinity of ordinary second order differential equations. In this case, the zeroth order solution is that due to Mangler, while the higher order approximations represent the corrections arising from the increased transverse curvature effect over that introduced in the zeroth approximation.

The class of bodies considered in the present analysis is of the form $r_0 = ax^n$ and $r_0 = ae^{bx}$, (a , b and n are positive constants). The cylinder ($n=0$) and cone ($n=1$) are seen to be special cases in which the assumption of zero pressure gradient is justified* for all Mach numbers for the cylinder, and for supersonic flow for the cone. Although the equations for the other body shapes are given, only these two cases have been solved numerically to the point where the skin friction and heat transfer coefficients are evaluated. For both the cone and cylinder only the first order or Δ/r_0 correction to the Mangler formulation has been obtained, and then only for the case where the Prandtl number has been taken equal to unity. Finally, the problems remaining for future investigation are outlined.

2. THEORY

2.1 Boundary Layer Equations for Axial Symmetry and Particular Integrals of the Energy Equation for $Pr = 1$

In the present work it is assumed that the specific heat and Prandtl number are constant, the body forces are negligible, and that the gas obeys the perfect gas law

$$(3) \quad p = \rho R T$$

Here, p is the static pressure, ρ the mass density, T the absolute temperature, and R the gas constant per gram. As pointed out previously, the usual assumption of negligible longitudinal curvature of the meridian profile, that is δK_1 and $\delta^2 \frac{dK_1}{dx}$ small compared to unity is still made.

* At least idealistically within the usual boundary layer approximations. This point will be considered later in the paper.

However, the transverse curvature effect present in the axially-symmetric flow is considered, so that δK_2 can be of order unity. Following the usual order of magnitude analysis, the axially-symmetric boundary layer equations become:
equation of continuity,*

$$(4) \quad \frac{\partial}{\partial x} (\rho r u) + \frac{\partial}{\partial y} (\rho r v) = 0$$

momentum equation,

$$(5) \quad \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{dp}{dx} + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) + \frac{\mu}{r} \frac{\partial r}{\partial y} \frac{\partial u}{\partial y}$$

energy equation,

$$(6) \quad \rho \left(u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} \right) = u \frac{dp}{dx} + \frac{1}{\rho r} \left[\frac{\partial}{\partial y} \left(\mu \frac{\partial h}{\partial y} \right) + \frac{\mu}{r} \frac{\partial r}{\partial y} \frac{\partial h}{\partial y} \right] + \mu \left(\frac{\partial u}{\partial y} \right)^2$$

Here, u and v are the velocity components in the x and y directions respectively, and $h = c_p T$ is the enthalpy of the gas where c_p is the specific heat at constant pressure. The Prandtl number, $Pr = c_p \mu / k$ where μ is the coefficient of viscosity, and k is the coefficient of thermal conductivity of the gas. It is to be noted that the static pressure across the boundary layer is still found to be constant to our order of approximation.

* There is some question regarding the generality of the continuity equation in this form, however, so far as the present paper is concerned it is justifiable to write it in this way.

The boundary conditions on the velocity follow from continuity and the requirement of no slip at the wall. The temperature may satisfy the condition that there is no heat transfer at the wall, or the surface temperature may be specified. Therefore at

$$\begin{aligned}
 (7a) \quad y = 0 \quad u = v = 0 \\
 \quad \quad \quad h = h_w \quad \text{non-insulated case} \\
 \quad \quad \quad \text{or} \quad \frac{\partial h}{\partial y} = 0 \quad \text{insulated case}
 \end{aligned}$$

At infinite normal distance from the surface, or the "edge" of the boundary layer* the values of u and T are specified, so that for

$$\begin{aligned}
 (7b) \quad y = \delta \quad u = u_e \\
 \quad \quad \quad h = h_e
 \end{aligned}$$

Here, the subscript e is used to denote the inviscid flow values so that in the case of the cone, for example, this would represent the conditions on the downstream side of the conical shock. Of course, for zero pressure gradient all the inviscid quantities, u_e , h_e , p_e , etc..., are constant.

If the Prandtl number of the gas is equal to unity, then a most interesting result is obtained by multiplying the momentum equation by u and adding this product to the energy equation, to give

$$\rho u \frac{\partial}{\partial x} \left(h + \frac{1}{2} u^2 \right) + \rho v \frac{\partial}{\partial y} \left(h + \frac{1}{2} u^2 \right) = \frac{\mu}{T} \frac{\partial T}{\partial y} \frac{\partial}{\partial y} \left(h + \frac{1}{2} u^2 \right) + \frac{\partial}{\partial y} \left[\mu \frac{\partial}{\partial y} \left(h + \frac{1}{2} u^2 \right) \right]$$

* For a discussion of this point see p. 381 of Ref. 7 as an example.

A particular integral of this equation is given immediately by

$$(8) \quad h + 1/2 u^2 = \text{constant}$$

This is of course the well known Buscmann integral,¹⁷ which is valid for an isoeenergetic free stream and an insulated surface, regardless of the pressure gradient. It says that the energy per unit mass in the boundary layer is a constant, and therefore, at least in the case of $Pr = 1$, the transverse curvature effects leave the recovery temperature unchanged. This conclusion is true for all values of Δ/r_0 , and its significance can best be illustrated by noting that the recovery temperature on an insulated needle, for $Pr = 1$, will be the same as on any insulated two-dimensional airfoil.

In the case of zero pressure gradient, it is further found that

$$h = A + B u + 1/2 u^2$$

is a particular integral of the boundary layer momentum and energy equations. If the boundary conditions of constant surface and free stream temperature are applied to the evaluation of the constants A and B, then the above relation can be written as,

$$(9) \quad \frac{T}{T_e} = \frac{T_w}{T_e} + \left[\left(1 + \frac{\gamma-1}{2} M_e^2 \right) - \frac{T_w}{T_e} \right] \left(\frac{u}{u_e} \right) + \frac{\gamma-1}{2} M_e^2 \left(\frac{u}{u_e} \right)^2$$

This integral is the well known result first found by Crocco¹⁸ in connection with the flow over a flat plate with constant surface temperature. Therefore, for $Pr = 1$ one can conclude that on an axially-symmetric body for all Δ/r_0 the transverse curvature effect will not alter the

form of the temperature distribution through the boundary layer from that of a flat plate. In other words the significance of this result can be summarized by noting that, at least for $Pr \approx 1$, the Reynolds analogy parameter which is proportional to the ratio of the surface heat transfer to the skin friction, will be the same on a needle as on a flat plate.

2.2. Reduction of Equations to Almost Two Dimensional Form and Physical Interpretation of Transverse Curvature Terms

If $\Delta/r_0 \ll 1$, then the transverse curvature terms, $\frac{\mu}{r} \frac{\partial r}{\partial y} \frac{\partial u}{\partial y}$, $\frac{\mu}{r} \frac{\partial r}{\partial y} \frac{\partial h}{\partial y}$, can be neglected in both the momentum and energy equations, so that they then assume the two-dimensional form. This is tantamount to replacing $r(x, y)$ by $r_0(x)$, which, if carried out in the continuity equation reduces it to

$$\frac{\partial}{\partial x} (\rho r_0 u) + r_0 \frac{\partial}{\partial y} (\rho v) = 0$$

With the compressible axially-symmetric boundary layer equations in the above form, Mangler⁸ was able to transform them to those of a two-dimensional flow. He accomplished this by a change of independent variables governed by the relations

$$(d\bar{x})_m = \frac{r_0^2}{L^2} dx, \quad (d\bar{y})_m = \frac{r_0}{L} dy$$

where L is a characteristic fixed reference length. A suitable redefinition of the dependent variables was also required; the reader is referred to Mangler's original papers.

In the present analysis, because Δ/r_0 may approach the order of unity, it is not possible to replace $r(x, y)$ by $r_0(x)$, and hence to reduce

the axially-symmetric boundary layer equations to a two-dimensional form by Mangler's transformation. However, by a simple generalisation of this transformation, the equations can be put into a nearly-two-dimensional form.

The transformation of independent variables from x, y to \bar{x}, \bar{y} is made by means of the relations

$$(10) \quad d\bar{x} = \frac{r_0^2}{L^2} dx \quad d\bar{y} = \frac{r(x, y)}{L} dy$$

Here, it can be seen that the x -coordinate transformation is the same as that given by Mangler, but in the change of the independent variable y , $r_0(x)$ is replaced by $r(x, y)$. A geometrical interpretation of the normal coordinate transformation follows from the fact that

$$\begin{aligned} \bar{y} &\sim \int r(x, y) dy = \frac{\int r \cos \alpha dy}{\cos \alpha} \\ &= (\text{transverse viscous area}) \sec \alpha \end{aligned}$$

In other words, the coordinate \bar{y} is proportional to the boundary layer area in the transverse plane, (see Fig. 1), projected onto a plane normal to the body surface. On the other hand

$$\bar{x} \sim \int r_0^2 dx$$

so that the coordinate \bar{x} is proportional to the volume swept out by the body. Now, the transverse curvature effects are associated with the circumferential spreading of the viscous layer. Therefore, the rate at

which the body circumference changes with length will be the axially-symmetric geometrical factor determining the characteristics of the boundary layer. It is clear that the \bar{x} coordinate which is the distorted distance along the surface, essentially characterizes the overall geometry of the body while the \bar{y} coordinate involves the resultant transverse viscous curvature effect. Therefore, it is not surprising that in attempting to reduce the equations to a near two-dimensional form, the \bar{x} transformation should be the same as given by Mangler. On the other hand Mangier replaced every point in the boundary layer by the corresponding surface point. Therefore, the "corresponding" projected viscous area in Mangler's case would be only a first approximation to the "proper" value given above.

Using Eqs. (10) the following transformation formulae are obtained,

$$(11) \quad \frac{\partial}{\partial x} = \frac{r_0^2}{L^2} \frac{\partial}{\partial \bar{x}} + \frac{\partial \bar{y}}{\partial x} \frac{\partial}{\partial \bar{y}}$$

$$\frac{\partial}{\partial y} = \frac{r}{L} \frac{\partial}{\partial \bar{y}}$$

where it is not necessary to evaluate $\frac{\partial \bar{y}}{\partial x}$. As indicated in Section 1.1 from the body geometry, $r = r_0(x) + y \cos \alpha$, so that

$$(a) \quad \bar{y} = \int \frac{r}{L} dy = \frac{r_0 y}{L} + \frac{y^2 \cos \alpha}{2L}$$

from which,

$$(12) \quad b) \quad \frac{r^2}{r_0^2} = 1 + \frac{2L \cos \alpha}{r_0^2} \bar{y}$$

and

$$(c) \quad \frac{\partial}{\partial \bar{y}} \left(\frac{r^2}{r_0^2} \right) = \frac{2L \cos \alpha}{r_0^2}$$

If a new velocity \bar{v} is defined by,

$$(13) \quad \bar{v} = \frac{rL}{r_o^2} v + \frac{L^2}{r_o^2} \frac{\partial \bar{y}}{\partial x} u$$

then from Eqs. (11) and (12) the boundary layer equations [Eqs. (4) - (6)] become,

equation of continuity,

$$(14) \quad \frac{\partial}{\partial \bar{x}} (\rho u) + \frac{\partial}{\partial \bar{y}} (\rho \bar{v}) = 0$$

momentum equation,

$$(15) \quad \rho \left(u \frac{\partial u}{\partial \bar{x}} + \bar{v} \frac{\partial u}{\partial \bar{y}} \right) - \frac{\partial}{\partial \bar{y}} \left(\mu \frac{\partial u}{\partial \bar{y}} \right) + \frac{dp}{d\bar{x}} = \frac{2L \cos \alpha}{r_o^2} \frac{\partial}{\partial \bar{y}} \left(\bar{y} \mu \frac{\partial u}{\partial \bar{y}} \right)$$

energy equation,

$$(16) \quad \rho \left(u \frac{\partial h}{\partial \bar{x}} + \bar{v} \frac{\partial h}{\partial \bar{y}} \right) - \frac{1}{Pr} \frac{\partial}{\partial \bar{y}} \left(\mu \frac{\partial h}{\partial \bar{y}} \right) - \mu \left(\frac{\partial u}{\partial \bar{y}} \right)^2 - u \frac{dp}{d\bar{x}} \\ = \frac{2L \cos \alpha}{r_o^2} \left[\frac{1}{Pr} \frac{\partial}{\partial \bar{y}} \left(\mu \bar{y} \frac{\partial h}{\partial \bar{y}} \right) + \mu \bar{y} \left(\frac{\partial u}{\partial \bar{y}} \right)^2 \right]$$

The continuity equation is now in a two-dimensional form of a compressible flow in the \bar{x} , \bar{y} plane with velocity components u and \bar{v} respectively. The left hand sides of both the momentum and energy equations are also two-dimensional in form. Therefore, the non-two-dimensional terms on the right hand side of the momentum and energy equations must carry the burden of the increased transverse curvature effect over that which is obtained using the Mangler formulation. For simplicity we will consider only the additional shear term in the momentum equation in making a qualitative examination of the changes

wrought by these terms, since analogous conclusions will hold for the added heat flux and dissipation terms in the energy equation.

In the present paper, we are concerned with the case where in zeroth approximation the effect of the added shear term can be neglected, so that any changes which arise can be considered essentially perturbations in Δ/r_0 on the Mangler flow. In this case, as will be shown, the added shear term behaves like a pressure gradient in two-dimensional flow. For the time being, without going very deeply into the underlying reasons, this result can be seen clearly through two analogies.

The first analogue is the so-called "weak interaction" self-induced pressure gradient which is generated in hypersonic flow,² as the result of the interaction of the longitudinal curvature of the viscous layer with the external flow. In this case the effects produced by the self-induced pressure gradient are essentially perturbations superposed on an already existing uniform flow. In the present problem there is an analogous "interaction" of the circumferential growth of the viscous layer, not with any external flow, but rather with the shear pattern obtained by considering the effect of the rate of change of circumference to be small, or even absent. Thus, it is a phenomenon perturbed with respect to the transverse curvature, Δ/r_0 , making it a purely "axial" effect. The second analogue stems from comparisons of the additional shear term to the modified effective pressure gradient term which arises as a result of transforming the normal coordinate in a planar compressible flow by means of Howarth's¹⁹ relation, in order to reduce the equations to an incompressible form, (see Ref. 7, pp. 434-436).

Accepting that the added shear term can be considered to manifest itself as a pressure gradient, then the results obtained by previous authors become clear. Now, it must be emphasized that it is not at all necessary to utilize this pressure gradient analogy to show the direction of such general quantities as the skin friction. This could follow directly and perhaps even more simply from considerations of the three-dimensional nature of the problem, such as the retarding force per unit area when compared with that of a flat plate. However, it is used because it does show up clearly not only the direction but also the form of many of the results, at least in the region where Δ/r_0 is less than unity. For example, in the original work on entry flow in a cylinder by Atkinson and Goldstein,⁶ it explains why the self-induced longitudinal pressure gradient appears in the equations in exactly the same manner as the transverse curvature effect.* In the work of Seban and Bond¹³ on the incompressible flow over a cylinder, it answers why the wall shear should increase. This follows from the fact that since the transverse curvature term is always positive, then it behaves like a favorable pressure gradient which would tend to increase the skin friction coefficient.⁶ A favorable gradient will also increase the heat transfer coefficient^{21, 22, 23} but as is well known from planar calculations

*This same property shows up from work by the present authors²⁰ on the self-induced pressure gradient generated in the hypersonic viscous flow over a cone.

the change in recovery factor is small²¹⁻²³. As one might expect these same results were obtained by Seban and Bond in their calculations. The smallness of the numerical change in recovery factor suggests that actually the change might be due to additive numerical errors, whereas mathematically the change is in fact zero. This was shown by Probstein and Lees²⁴ for the pressure gradient generated by the hypersonic induced effect over a flat plate in the weak-interaction region. Finally, the fact that they found the boundary layer displacement thickness to be only slightly reduced in comparison with the flat plate value is a result which is also to be expected. This implies that most of the changes in velocity must therefore occur relatively close to the cylinder.

Turning to the work of Cooper and Tulin¹⁵ it now is clear why they found that the increase in wall shear on the cylinder for favorable gradients, when compared with the shear on the flat plate for the same favorable gradient, is less than the increase on the cylinder in uniform flow when compared with the flat plate in uniform flow. The converse is true for adverse gradients. In other words, what their results say, (see Fig. 4, Ref. 15), is that for a given value of Δ/r_0 ,

$$\left(\frac{\tau_{\text{cyl. fav. grad.}}}{\tau_{\text{cyl. adv. grad.}}} \right) \left(\frac{\tau_{\text{f-p same adv. grad.}}}{\tau_{\text{f-p same fav. grad.}}} \right) > 1$$

But, the wall shear for planar flow with an adverse gradient is less than the wall shear with a favorable gradient. Since the product shown above must be greater than unity, it follows that at a given value of Δ/r_0 , the wall shear on the cylinder in a favorable gradient is greater than the

wall shear for a cylinder in an adverse gradient. Certainly, this result is obvious although it was not explicitly pointed out in Ref. 15; that is $-\frac{dp}{dx}$ and $\frac{2L \cos \alpha}{r_c} \frac{\partial}{\partial y} (\bar{y} \tau)$ add when the gradient is favorable and subtract when the gradient is adverse. This interesting problem of the "interaction" of the pressure gradient term with the transverse curvature shear term which manifests itself like a favorable pressure gradient, is reserved for a forthcoming paper. Additional conclusions which can be drawn about the present work once it is recognized that the added shear term behaves like a pressure gradient, will be reserved until the equations are obtained in the form in which they will be solved.

2.3 The Incompressible Plane and Similarity Considerations

The boundary layer equations now being in a nearly-two-dimensional form, suggest first a transformation from the compressible to an incompressible form. However, before attempting to do this two assumptions are made; the first is that the viscosity varies linearly with the temperature, while the second one is that the pressure gradient is taken to be zero.

Following Chapman²⁵ a parameter C_μ is introduced such that

$$\frac{\mu}{\mu_c} = C_\mu \frac{T}{T_c}$$

Since the Prandtl number and specific heat are constant, the heat conduction coefficient varies in the same manner as the coefficient

of viscosity, that is

$$\frac{\mu}{\mu_e} = \frac{\mu_w}{\mu_e}$$

The constant C_e can be determined by matching the viscosity relation with the theoretically determined value (e.g. Hirschfelder et al²⁶) at the wall temperature, so that

$$C_e = \frac{\left(\frac{\mu_w}{\mu_e}\right)}{\left(\frac{T_w}{T_e}\right)}$$

When a semi-empirical relation such as Sutherland's equation is employed, then

$$\frac{\mu}{\mu_e} = f(T, T_e)$$

and

$$C_e = \frac{f(T_w, T_e)}{\left(\frac{T_w}{T_e}\right)}$$

For $T_w \gg T_e$ one finds that $0 < C_e \leq 1$. As pointed out by Chapman and Rubesin²⁷ the above relation retains the advantages of the linear form while allowing for greater accuracy in the important region of the boundary layer flow near the surface rather than near the free stream.

Idealistically of course the condition of zero pressure gradient so that $p = \text{constant} = p_e$, is only realized in two cases; a cylinder with its generators parallel to the flow, and an unyawed cone in supersonic flow. In the case of a cylinder, there is considerable complication near the nose. Theoretically however, one can visualize a case in which suction through the interior of the cylinder makes the

stagnation stream surfaces coincide with the cylinder surface. This is supposed possible even in supersonic flow when there is a detached shock wave in front of the cylinder. The boundary layer is then supposed to have zero thickness at $x = 0$. Of course, as pointed out in the introduction, the overall flow problem is to be treated in such a way that all possible information about the nature of the flow downstream of the "immediate" nose region is obtained, which does not depend on the detailed history of the flow in this region.

It is well known for a cone in supersonic flow with an attached shock wave, that the values of all physical quantities are constant along the cone surface. As has already been pointed out however, other effects might arise from the propagation of the immediate nose influence downstream, or possibly from a self-induced pressure gradient. These and other phenomena which have not already been considered are not taken into account and this must be borne in mind when comparing the results of the present investigation with experimental data.

For all other body shapes and flow conditions, the pressure gradient is not zero. Actually however for certain body shapes and flight speeds the contribution of the pressure gradient might be of a higher order than the transverse curvature effect. Therefore once the body shape is known it may be possible to determine the order of the pressure gradient in comparison with the order of Δ/r_0 . The investigation of this problem will also be reserved for a forthcoming paper.

In order to transform the modified boundary layer equations [Eqs. (14)-(16)] to an incompressible form Howarth's transformation¹⁹ is used. It is defined by the change of independent variables \bar{x}, \bar{y} to \bar{x}, Y where

$$(17a) \quad Y = \int \frac{\rho}{\rho_e} d\bar{y}$$

so that

$$(17b) \quad \left(\frac{\partial}{\partial \bar{x}} \right)_{\bar{y}} = \left(\frac{\partial}{\partial \bar{x}} \right)_Y + \frac{\partial Y}{\partial \bar{x}} \frac{\partial}{\partial Y}$$

$$\frac{\partial}{\partial \bar{y}} = \frac{\rho}{\rho_e} \frac{\partial}{\partial Y}$$

As it turns out, it is more convenient to use the stream function ψ as the dependent variable. It is defined from the equation of continuity [Eq. (4)] which is satisfied immediately by writing

$$(18) \quad \frac{\rho}{\rho_e} + u \frac{\partial}{\partial \bar{y}} = \frac{\partial \psi}{\partial \bar{y}} \quad \text{and} \quad \frac{\rho}{\rho_e} + v = - \frac{\partial \psi}{\partial \bar{x}}$$

Since the present paper is concerned with a two-dimensional analogue, it is more appropriate to let

$$\bar{\Psi} = \psi/L$$

By means of Eqs. (17) and (18) the transformation of the boundary layer momentum and energy equations to the almost two-dimensional incompressible \bar{x} - Y plane gives for the momentum equation,

$$(19) \quad \frac{\partial \bar{\Psi}}{\partial Y} \frac{\partial^2 \bar{\Psi}}{\partial \bar{x} \partial Y} - \frac{\partial \bar{\Psi}}{\partial \bar{x}} \frac{\partial^2 \bar{\Psi}}{\partial Y^2} - c_{pe} \frac{\partial^3 \bar{\Psi}}{\partial Y^3} = \frac{2L c_{pe} \cos \alpha}{r_0^2} \frac{\partial}{\partial Y} \left[\frac{\partial \bar{\Psi}}{\partial Y^2} \left(\frac{\rho_e}{\rho} dY \right) \right]$$

and for the energy equation

$$(20) \quad \frac{\partial \bar{\Psi}}{\partial Y} \frac{\partial h}{\partial \bar{x}} - \frac{\partial \bar{\Psi}}{\partial \bar{x}} \frac{\partial h}{\partial Y} - \frac{c_{pe}}{Pr} \frac{\partial^2 h}{\partial Y^2} - c_{pe} \left(\frac{\partial^2 \bar{\Psi}}{\partial Y^2} \right)^2 = \frac{2L c_{pe} \cos \alpha}{r_0^2} \left[\frac{1}{Pr} \frac{\partial}{\partial Y} \left(\frac{\partial h}{\partial Y} \left(\frac{\rho_e}{\rho} dY \right) \right) + \frac{\partial^2 \bar{\Psi}}{\partial Y^2} \left(\frac{\rho_e}{\rho} dY \right) \right]$$

Illingworth²⁸ has examined the problem of the conditions under which similar velocity and temperature distributions for different values of x can be found in compressible planar flow. He concluded that such solutions only exist if the external velocity is constant, and then as in incompressible flow the similarity variable is of the form y/\sqrt{x} . Since to the approximation of Mangler the axially-symmetric compressible boundary layer equations can be put into a two-dimensional form, then it would appear logical to seek solutions in a variable proportional to y/\sqrt{x} . That is, the Mangler result would provide the so-called zeroth order solution for the present analysis. However, even to the approximation of Mangler, although the velocity and temperature distributions are derivable from ordinary differential equations, these distributions are not similar in the strict sense. It would nevertheless also be interesting to determine under what conditions if any, "pseudosimilar" profiles might be obtained without any approximations in the boundary layer equations other than those already made. If a similarity variable η is defined as

$$(21a) \quad \eta = \left(\frac{\mu_e}{\epsilon_e \gamma_e \bar{x}} \right)^{1/2} \gamma$$

then the transformation equations are given by

$$(21b) \quad \left(\frac{\partial}{\partial \bar{x}} \right)_\gamma = \left(\frac{\partial}{\partial \bar{x}} \right)_\eta + \frac{\partial \eta}{\partial \bar{x}} \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial \gamma} = \left(\frac{\mu_e}{\epsilon_e \gamma_e \bar{x}} \right)^{1/2} \frac{\partial}{\partial \eta}$$

From the above relations the momentum and energy equations

[Eqs. (19) and (20)] become respectively,

$$(22) \quad \frac{\partial \bar{\Psi}}{\partial \eta} \frac{\partial^2 \bar{\Psi}}{\partial \bar{x} \partial \eta} - \frac{\partial \bar{\Psi}}{\partial \bar{x}} \frac{\partial^2 \bar{\Psi}}{\partial \eta^2} - \frac{1}{2\bar{x}} \left(\frac{\partial^2 \bar{\Psi}}{\partial \eta^2} \right)^2 - \left(\frac{\epsilon_e \gamma_e \mu_e}{\bar{x}} \right)^{1/2} \frac{\partial^2 \bar{\Psi}}{\partial \eta^2} = \frac{2L\epsilon_e \gamma_e \cos \alpha}{r_0^2} \left[\frac{\partial^2 \bar{\Psi}}{\partial \eta^2} \right] \left[\frac{T}{T_e} d\eta \right]$$

and

$$(23) \quad \frac{\partial \bar{\Psi}}{\partial \eta} \frac{\partial h}{\partial \bar{x}} - \frac{\partial h}{\partial \eta} \frac{\partial \bar{\Psi}}{\partial \bar{x}} - \left(\frac{\epsilon_e \gamma_e \mu_e}{\bar{x}} \right)^{1/2} \frac{1}{Pr} \frac{\partial^2 h}{\partial \eta^2} - \epsilon_e \gamma_e \left(\frac{\mu_e}{\epsilon_e \gamma_e \bar{x}} \right)^{3/2} \left(\frac{\partial^2 \bar{\Psi}}{\partial \eta^2} \right)^2$$

$$= \frac{2L\epsilon_e \gamma_e \cos \alpha}{r_0^2} \left\{ \frac{1}{Pr} \frac{\partial}{\partial \eta} \left[\frac{\partial h}{\partial \eta} \right] \left[\frac{T}{T_e} d\eta \right] + \left(\frac{\mu_e}{\epsilon_e \gamma_e \bar{x}} \right) \left(\frac{\partial^2 \bar{\Psi}}{\partial \eta^2} \right)^2 \left[\frac{T}{T_e} d\eta \right] \right\}$$

With the equations in this form we may now inquire as to the condition for the existence of "similar" profiles or more correctly the requirement for the reduction of these partial differential equations to ordinary differential equations.

In general one can write,

$$(24a) \quad \psi = L (\epsilon_e \gamma_e \mu_e)^{1/2} \sqrt{\bar{x}} f(\bar{x}, \eta)$$

and

$$(24b) \quad \frac{T}{T_e} = \lambda(\bar{x}, \eta)$$

However if it is assumed that say, f and λ are functions of η alone, then on substitution of the above relations both the momentum and energy equations should involve only the independent variable η . If this is done, the left hand sides of both Eqs. (22) and (23) are in fact found to be dependent only upon η , but the right hand sides involve a function of η multiplied by a function of x . Therefore for pseudo-similarity to exist the body must have a shape such that this function of x , given by

$$\sqrt{\frac{C \mu_e}{\mu_e}} \frac{L \cos \alpha \sqrt{\bar{x}}}{r_0^2}$$

is a constant. Since $\sin \alpha = \frac{dr_0}{dx}$, then the criterion for the body shape is the following ordinary integro-differential equation in r_0 .

$$\left[1 - \left(\frac{dr_0}{dx} \right)^2 \right] \int_0^x r_0^2 dx = \text{constant } r_0^4$$

It is understood that the equation which Mark¹⁰ has examined, is the ordinary differential equation for the velocity distribution in incompressible flow which results when the above relation is identically satisfied. Here, no approximations regarding the order of Δ/r_0 have been made, so that the ordinary differential equations which result are valid for both the nose and downstream regions. The relation describing the body shape is non-linear and no integral of it

*See Eq. (26) with $\bar{\xi} = \text{constant}$, and $T = T_p = \text{constant}$.

has been found. If however, the body is assumed sufficiently slender so that $\alpha^2 = \left(\frac{dr_0}{dx}\right)^2$ and higher order terms can be neglected,* then the equation becomes

$$\int r_0^2 dx = \text{constant } r_0^4$$

This can be integrated immediately to give $r_0 = \text{constant } \sqrt{x}$. To the order of approximation being considered, this is the equation of a paraboloid. This same result can be obtained by assuming at the outset that the boundary layer equations can be written in the cylindrical polar form used in analyzing the flow over a cylinder.

Returning to the general case, f and λ are functions of the two variables \bar{x} and η . As noted already in great detail, in the present paper we are concerned primarily with the downstream region where Δ/r_0 is less than or possibly of the order of unity. Logically therefore, in order to solve these non-linear partial differential equations in such a region both f and λ could be expanded in asymptotic series in powers of a parameter $\xi \sim \Delta/r_0$, where the coefficients are functions of η alone and ξ is small in comparison to unity, (see Eq. (2)). The natural coordinate system for these equations would therefore be ξ and η . In the Mangler region, which is after all a part of the downstream region, one finds that the boundary layer thickness on bodies of revolution under zero pressure gradient is given by,

*As noted in a footnote in Sect. 1.1, the approximation is excellent for the regions being considered.

$$\delta \sim \frac{1}{r_0} \left\{ \int r_0^2 dx \right\}^{1/2}$$

Therefore a parameter

$$\xi \sim \frac{\Delta}{r_0} \sim \frac{\delta \cos \alpha}{r_0} \sim \frac{\cos \alpha}{r_0^2} \left\{ \int r_0^2 dx \right\}^{1/2}$$

is chosen, which when properly non-dimensionalized is given by

$$(25) \quad \xi = \frac{\sqrt{c \nu_e}}{u_e} \frac{L \cos \alpha \sqrt{\bar{x}}}{r_0^2}$$

By making a final transformation of coordinates from \bar{x}, η to ξ, η

the momentum and energy equations become respectively,

$$(26) \quad \xi \left(\frac{1}{2} - \mathcal{K} \right) \left[\frac{\partial f}{\partial \eta} \frac{\partial f}{\partial \xi \partial \eta} - \frac{\partial f}{\partial \xi} \frac{\partial^2 f}{\partial \eta^2} \right] - \frac{1}{2} f \frac{\partial^2 f}{\partial \eta^2} - \frac{\partial^3 f}{\partial \eta^3} = 2 \xi \frac{\partial}{\partial \eta} \left[\frac{\partial^2 f}{\partial \eta^2} \int \lambda d\eta \right]$$

and

$$(27) \quad \xi \left(\frac{1}{2} - \mathcal{K} \right) \frac{\partial f}{\partial \eta} \frac{\partial \lambda}{\partial \xi} - \left[\frac{1}{2} f + \left(\frac{1}{2} - \mathcal{K} \right) \xi \frac{\partial f}{\partial \xi} \right] \frac{\partial \lambda}{\partial \eta} - \frac{1}{Pr} \frac{\partial^2 \lambda}{\partial \eta^2} - (\gamma - 1) M_e^2 \left(\frac{\partial^2 f}{\partial \eta^2} \right)^2 \\ = 2 \xi \left\{ \frac{1}{Pr} \frac{\partial}{\partial \eta} \left[\frac{\partial \lambda}{\partial \eta} \int \lambda d\eta \right] + (\gamma - 1) M_e^2 \left(\frac{\partial^2 f}{\partial \eta^2} \right)^2 \int \lambda d\eta \right\}$$

where

$$(28) \quad \mathcal{K} = \left\{ \frac{\bar{\alpha}}{r_0} + \frac{r_0''}{(1 - r_0'^2)} \right\} \frac{r_0'}{r_0^2} \int r_0^2 dx$$

It must be emphasized that within the boundary layer approximations

these equations are valid for all values of Δ/r_0 .

Of interest is the fact that when $\xi = \text{constant}$ the implication from either the asymptotic expansions or the foregoing

equations is that f and λ are functions of η alone. But $\xi = \text{constant}$ is just the integro-differential equation describing the "near paraboloid" which was found previously as the criterion for the reduction to ordinary differential equations. In other words, when the boundary layer thickness goes like the body radius similarity in its restricted meaning is possible.

2.4 Summary of Mathematical Development

At this point, it seems suitable to sum up the transformations and assumptions that have been made. Thus for a perfect gas with constant specific heat and Prandtl number, under the assumptions of a linear viscosity-temperature law and zero pressure gradient, a transformation of coordinates has been made from x, y to ξ, η where

$$\xi(x) = \sqrt{\frac{\mu_e \gamma_e}{\mu_e}} \frac{\cos \alpha \left\{ \int r_0^2 dx \right\}^{1/2}}{r_0^2}$$

$$\eta(x, y) = \sqrt{\frac{\mu_e}{\mu_e \gamma_e}} \frac{1}{\left\{ \int r_0^2 dx \right\}^{1/2}} \int \frac{f}{\rho_e} r dy$$

The transformation equations are given by

$$\frac{\partial}{\partial x} = \frac{d\xi}{dx} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta}$$

and

$$\frac{\partial}{\partial y} = \sqrt{\frac{\mu_e}{\mu_e \gamma_e}} \frac{1}{\left\{ \int r_0^2 dx \right\}^{1/2}} \frac{\rho_e r}{\rho_e} \frac{\partial}{\partial \eta}$$

where in this case it is not necessary to evaluate $\frac{\partial \eta}{\partial x}$.

A stream function ψ which satisfies the equation of continuity identically, is defined by

$$\frac{\rho}{\rho_e} + u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\rho}{\rho_e} + v = - \frac{\partial \psi}{\partial x}$$

Writing

$$\psi = \sqrt{C_e r_e u_e} \left\{ \int_0^{\eta} r_0^2 dx \right\}^{1/2} f(\xi, \eta)$$

gives
$$\frac{u}{u_e} = \frac{\partial f(\xi, \eta)}{\partial \eta}$$

The static temperature ratio which is also a function of ξ and η is written

$$\frac{T}{T_e} = \lambda(\xi, \eta)$$

The partial differential equations defining $f(\xi, \eta)$ and $\lambda(\xi, \eta)$ valid for all values of Δ/r_0 within the boundary layer approximations are given by Eqs. (26) and (27).

3. SOLUTION OF EQUATIONS

3.1 Asymptotic Expansions, Boundary Conditions, and Zeroth Order Equations

As noted previously, f and λ are to be expanded in asymptotic series in powers of the parameter ξ , where the coefficients are functions of η alone, that is

$$(29a) \quad f(\xi, \eta) = f_0(\eta) + \xi f_1(\eta) + \xi^2 f_2(\eta) + \dots = \sum_{j=0}^{\infty} \xi^j f_j(\eta)$$

and

$$(29b) \quad \lambda(\xi, \eta) = \lambda_0(\eta) + \xi \lambda_1(\eta) + \xi^2 \lambda_2(\eta) + \dots = \sum_{j=0}^{\infty} \xi^j \lambda_j(\eta)$$

Here, the coefficients are of order unity and the expansion parameter

ξ is assumed to be small in comparison to unity. It will be seen, that the coefficients f_j and λ_j for $j \geq 1$, will be functions of γ alone only so long as the body shapes which are considered fall into certain prescribed classes. It is of course these prescribed shapes with which the present paper is concerned.

The boundary conditions follow from Eqs. (7a) and (7b) as

$$(30a) \quad \begin{aligned} f_j(0) = f'_j(0) = 0 & \quad \text{for } j \geq 0 \\ f'_0(\infty) = 1, \quad f'_j(\infty) = 0 & \quad \text{for } j \geq 1 \end{aligned}$$

and

$$(30b) \quad \begin{aligned} \lambda_0(0) = \lambda_w, \quad \lambda_j(0) = 0 & \quad j \geq 1 \quad \text{non-insulated boundary} \\ \lambda'_j(0) = 0 & \quad j \geq 0 \quad \text{insulated boundary} \\ \lambda_0(\infty) = 1, \quad \lambda_j(\infty) = 0 & \quad \text{for } j \geq 1 \quad \text{both cases} \end{aligned}$$

Substituting the asymptotic expansions into Eqs. (26) and (27) and equating to zero all terms with the same power of ξ , a double infinity of ordinary differential equations is obtained. All these equations except for the zeroth order momentum equation, are found to be linear. In the present paper, only the zeroth and first order equations are considered, but the methods of solution can be extended to higher orders if necessary.

In zeroth order, the momentum equation is given by

$$(31a) \quad \partial f_0''' + f_0 f_0'' = 0$$

with boundary conditions $f_0(0) = f_0'(0) = 0$, and $f_0'(\infty) = 1$. This is of course the well known Blasius equation, the solution of which may be found tabulated by Howarth,²⁹ as well as in other standard works. It is to be noted that to this order the Blasius equation describes the flow for all bodies, and the shape does not enter the problem except as it is prescribed in the coordinate transformation which in this approximation is given by Mangler. Of course, such a result is to be expected since it has been assumed that the pressure gradient is zero.

For the energy equation, the zeroth order relation is given by

$$(31b) \quad \frac{1}{Pr} \lambda_0'' + \frac{1}{2} f_0 \lambda_0' + (\gamma-1) M_e^2 f_0''^2 = 0$$

with the boundary conditions $\lambda_0(0) = \lambda_w$ for heat transfer or $\lambda_0'(0) = 0$ with an insulated wall and $\lambda_0(\infty) = 1$. For $Pr = 1$ the complete analytic solution of the energy equation has already been given [Eq. (9)], while for $Pr \neq 1$ the numerical solution has been tabulated by Crocco³⁰ for various values of the Prandtl number.

3.2 First Order Equations and Admissable Body Classes

On carrying out the prescribed substitution of the asymptotic expansions, the first order momentum equation is found to be

$$(32) \quad f_1''' + \frac{1}{2} f_0 f_1'' - \left(\frac{1}{2} - \gamma\right) f_0' f_1' + (1 - \gamma) f_0'' f_1 = -2 (f_0'' \eta_0)'$$

where $\eta_0 = \int \lambda_0(\eta) d\eta$, and where the boundary conditions are $f_1(0) = f_1'(0) = f_1'(\infty) = 0$. Of course, \mathcal{K} which is defined by Eq. (28) must be equal to a constant in order that $f_1 = f_1(\eta)$, and it is this condition which prescribes the allowable body shapes in the present analysis. Again however, the resultant integro-differential equation cannot be integrated except when $r_0'' = 0$ which corresponds to the cone and cylinder, unless we limit ourselves to sufficiently slender bodies. For the sharp nosed slender bodies being considered, the slope α can be supposed small so that terms of order α^2 and higher might be neglected, and

$$\alpha \approx \sin \alpha = \frac{dr_0}{dx}$$

so that

$$\cos \alpha = (1 - r_0'^2)^{1/2} \approx 1$$

to this approximation. The longitudinal curvature K_1 is given by

$$K_1 = \frac{r_0''}{(1 - r_0'^2)} \approx r_0''$$

Note that the indicated differentiation is with respect to distance along the surface, and not along the axis. If K_1 is considered to be sufficiently small, then $(r_0'' / (1 - r_0'^2))$ is small when compared with $2/r_0$ and can be neglected, so that the equation $\mathcal{K} = \text{constant}$ reduces to

$$\mathcal{K} \approx \frac{2r_0' \int r_0'^2 dx}{r_0^3} = \text{constant}$$

As noted previously, in the case of the cone and cylinder the above relation is exact. Integration of this equation gives two mathematically

admissible classes of body shapes. The first is $r_0 = ax^n$ ($n \neq -1/2$), where the value of \mathcal{K} is given by $\mathcal{K} = \frac{2n}{2n+1}$ independently of a . Since r_0 must be finite at the origin then $n \geq 0$. One might note, that for the cylinder and cone, $\mathcal{K} = 0$, and $2/3$ respectively. The second class of bodies is $r_0 = ae^{bx}$ which corresponds to $\mathcal{K} = 1$ for all values of a and b . In any event the exponentially shaped open nosed bodies of revolution have doubtful applicability to aerodynamic problems, particularly under the assumption of zero pressure gradient. Nevertheless, as will be shown by analogy with the two-dimensional flow with pressure gradient, there is some question as to whether negative values of b even admit a solution to the problem.

The equation in f_1 [Eq. (32)] with different right hand sides, has occurred previously in the analysis of Howarth²⁹ on the problem of an incompressible boundary layer in two-dimensional flow under a linear pressure gradient. The equations f_1, f_2, \dots, f_8 occurring in Howarth's case correspond to values of $\mathcal{K} = -1/2, -3/2, -5/2, \dots, -15/2$ respectively. This mathematical similarity bears out the previous ideas regarding the fact that the transverse curvature effect manifests itself in a manner similar to a pressure gradient in a two-dimensional flow. If the analogy is carried further, it is immediately evident why the body shapes which were found, were of the form $r_0 = ax^n$ and $r_0 = ae^{bx}$. This follows from the fact that in the present problem the body radius, $r_0(x)$, replaces the external velocity, $u_e(x)$, of the two-dimensional

incompressible pressure gradient problem solved by Goldstein.³¹ He tried to determine the external velocity distributions which would give similar profiles. The only admissible solutions for the velocity were those found above for r_0 , and furthermore in the exponential case Goldstein felt that b had to be positive. This was verified shortly afterwards by Hardy.³²

Assuming the shapes to be such that $\mathcal{K} = \text{constant}$, then Eq. (32) is a third order linear ordinary differential equation in f_1 . The homogeneous equation is however only a perfect differential in the case of $\mathcal{K} = 1$, which corresponds to the exponential body shape. For all other values of \mathcal{K} , the equation has to be integrated by numerical methods, although it is possible to reduce its order by writing

$$(33) \quad f_1(\eta) = f_0' g(\eta) \quad \text{and} \quad g(\eta) = \bar{G}(\eta)$$

By use of the Blasius relation for f_0 the equation in f_1 is then reduced to the following second order linear equation in \bar{G} :

$$(34) \quad f_0' \bar{G}'' + (3f_0'' + \frac{1}{2}f_0 f_0') \bar{G}' + \left[f_0''' - (\frac{1}{2} - \mathcal{K}) f_0'^2 \right] \bar{G} = -2(f_0'' \eta_0)'$$

This equation can be numerically integrated once the value of \mathcal{K} has been selected. Integrations have been carried out for \bar{G} in the cases where $\mathcal{K} = 0$ and $2/3$, which correspond to the cylinder and cone respectively. The detailed methodology used will be presented in a forthcoming paper.

The first order energy equation is given by

$$(35) \quad \frac{1}{\rho_r} \lambda_1'' + \frac{1}{2} f_0 \lambda_1' - \left(\frac{1}{2} - \mathcal{X}\right) f_0' \lambda_1 = -\lambda_0' (1 - \mathcal{X}) f_1 - 2(\gamma - 1) M_\infty^2 f_0'' f_1'' - \frac{2}{\rho_r} (\lambda_0' \eta_0)' - 2(\gamma - 1) M_\infty^2 \eta_0 f_0''^2$$

where \mathcal{X} and η_0 are as defined previously. The boundary conditions are that $\lambda_1(0) = 0$ for the case of heat transfer, or $\lambda_1'(0) = 0$ for the insulated wall, while $\lambda_1(\infty) = 0$. For $Pr = 1$ the solution is known, but for $Pr \neq 1$ the equation must be integrated numerically by methods similar to those used in determining $\bar{G}(\eta)$.

3.3 Some Numerical Results for the Cone and Cylinder for $Pr = 1$

One of the primary quantities of interest is the wall shear, or skin friction coefficient c_f which is defined by,

$$c_f = \frac{\left(\mu \frac{\partial u}{\partial y}\right)_{y=0}}{\frac{1}{2} \rho_\infty u_\infty^2}$$

If the appropriate asymptotic expansion is substituted c_f can be shown to reduce to

$$(36) \quad c_f = 2 \sqrt{\frac{\tau_0 \gamma_e}{u_e}} \cdot \frac{\tau_0}{\left\{ \int_{\tau_0}^{\infty} dx \right\}^{1/2}} \int_{\tau_0}^{\infty} f''(b) db$$

Of equal importance is the local heat transfer rate which is defined by

$$q_w = - \left(k \frac{\partial T}{\partial y} \right)_{y=0}$$

In the case of $Pr = 1$, from the particular integral of the energy equation given by Eq. (9) it is simple to show that the local heat transfer rate is directly related to the skin friction coefficient (Reynolds analogy) by the following relation

$$(37a) \quad q = -\frac{1}{2} c_p \rho_e T_e u_e \left(1 + \frac{\gamma-1}{2} M_e^2 - \lambda_w \right) C_f$$

However, in general when $Pr \neq 1$ then

$$(37b) \quad q = -\frac{c_p}{Pr} \rho_e T_e \sqrt{C_e u_e v_e} \frac{\tau_w}{\left\{ \int \tau_e^2 dx \right\}^{1/2}} \sum_{j=0}^{\infty} \xi^j \lambda_j'(0)$$

To simplify the numerical work the Prandtl number is taken equal to unity since in that case, the Reynolds analogy parameter is constant and only the wall shear calculation need be carried out. Furthermore, only the first order correction (ξ or $j = 1$) is examined. The body shapes investigated were the cone and cylinder, which are after all the only cases where the assumption of zero pressure gradient can be theoretically justified.

For both the cone and the cylinder the results are presented as the ratio of the skin friction obtained by considering the transverse curvature effects not taken into account by Mangler, to the skin friction which one would get using the Mangler formulation. The calculated values were obtained by the numerical integration of Eq. (34), which gives for the cone

$$(38) \quad \frac{C_f}{(C_f)_M} = 1 + \xi \left[0.517 + 0.913 \lambda_w + 0.121(\gamma-1) M_e^2 \right] + \dots$$

where $\xi = \frac{1}{\sqrt{3} \tan \alpha} \sqrt{\frac{C_e}{(Re_x)_e}}$, and the subscript e represents the inviscid values downstream of the conical shock. Strictly speaking of course, the cone solution should only be valid for supersonic flow with an attached shock however, in Fig. 4 which is a graph of Eq. (38) for $\gamma = 1.4$, the values are carried down to $M_e = 0$ for continuity purposes. For the cylinder the first order equation governing the increase in wall shear is

$$(39) \quad \frac{C_f}{(C_f)_M} = 1 + \xi \left[0.684 + 1.407 \lambda_w + 0.173 (\gamma - 1) M_e^2 \right] + \dots$$

where $\xi = \frac{x}{r_0} \sqrt{\frac{C_e}{(Re_x)_e}}$, and the subscript e denotes values in the undisturbed free stream far from the body. A plot of Eq. (39) for $\gamma = 1.4$ is given in Fig. 5 and it is of interest to note, that the value of the ordinate parameter $\frac{r_0}{x} \sqrt{\frac{(Re_x)_e}{C_e}} \left(\frac{C_f - (C_f)_M}{(C_f)_M} \right)$ for $\lambda_w = 1$ and $M_e = 0$ is 2.091 which compares with the value of 2.12 given by Seban and Bond.

From Figs. 4 and 5 it can be seen that below a Mach number of about 3 the increase in skin friction coefficient on both the cone and cylinder for the heat transfer case is practically independent of the Mach number, although the dependency becomes significant around $M_e = 5$, and increasingly important for all higher Mach numbers. This is simply the manifestation of the increase in viscous dissipation associated with the higher flight speeds. It is also clear both from Figs. 4 and 5, and Eqs. (38) and (39), that the skin friction is larger at a constant ξ the higher the ratio of wall to free stream

temperature. Since $\bar{\xi}$ is proportional to Δ/r_0 , its constancy implies a constant value of the transverse curvature parameter. Finally, from Eqs. (38) and (39) it follows that for the same surface temperature ratio and Mach number on the cone and cylinder, at a constant value of the transverse curvature parameter, the skin friction increase will be larger on the cylinder than on the cone.

One of the more important results obtained from the calculation is the approximate range of validity of the solution. As was noted in Section 1.2 the ordinate of Fig. 3 when divided by $\sqrt{3}$ represents $\bar{\xi}$ for both the cone and cylinder. Thus, if the range where $\bar{\xi}$ is small compared to unity is considered then for $(\Delta/r_0)_M = 1, 0.75$, and 0.5 we have, by way of example, at $M_e = 4$ that $\bar{\xi}$ is $0.107, 0.081$, and 0.053 respectively. In the case of the insulated cone this gives values of $(c_f - c_{fM})/c_{fM} = 0.56, 0.42$, and 0.28 respectively. It would seem that in this case for $\Delta/r_0 = 1$ the change in c_f is somewhat too large to be given accurately by only the first term in the expansion. Nevertheless, what is clear is that for $\bar{\xi}$ less than about 0.1 , (somewhat less for the cylinder), which corresponds to Δ/r_0 in the range less than or of the order of unity, the present formulation would appear to be valid. It has already been shown that this region is in a practical range of interest. Therefore, for $\Delta/r_0 \gtrsim 1$ the increase in skin friction over what Mangler predicts can become important, and this change can be determined by the formulation given in the present paper. Of course for $Pr = 1$ the local heat transfer rate is directly

proportional to the skin friction, and is given by Eq. (37a). A final result which should be noted is that the displacement thickness for both the cone and cylinder is decreased from the Mangler value, but only slightly as expected.

4. FUTURE INVESTIGATIONS

One of the main problems for future investigation is a study of pressure gradients in axially-symmetric flow and their "interaction" with the transverse curvature phenomena. A study which would include both the hypersonic self-induced pressure gradient effect in addition to the transverse curvature effect would also be of considerable interest.

An investigation of the conditions under which the assumption of zero pressure gradient is justified is certainly required. It might be possible to determine this for the given body shape and flight speed by making a comparison of the order of the pressure gradient with the order of Δ/r_0 .

Since the calculations in the present paper have only been carried out to first order, it would be worthwhile to evaluate the O^2 contributions to increase the accuracy of the results and to establish more closely the range of validity of the present solutions. In addition, some numerical integrations of the energy equation for Prandtl numbers different from unity are needed, in order to determine the effect on the heat transfer rate and recovery factor. For special cases it might be possible to examine the recovery factor change analytically.

In the present investigation a linear viscosity-temperature relation has been assumed, so that to check its accuracy some calculations should be made in which a more realistic relation is utilized.

CONCLUSIONS

1. The viscous transverse curvature effect in axially-symmetric flow is characterized by the parameter Δ/r_0 , where Δ is the projection of the boundary layer thickness or more precisely the displacement thickness onto the transverse plane, and r_0 is the distance from any point on the body to the axis of symmetry.

2. There are two main asymptotic flow regions, which for a pointed body of revolution, where the radius increases with axial distance, can be represented by I, a "nose" region distinguished by the fact that $\Delta/r_0 \gg 1$, and II, a "downstream" region where Δ/r_0 is of the order of, or less than unity. The two regions are separated by a "transition" zone where Δ/r_0 is intermediate between these values.

3. The nose region is characterized by the fact that the stress term arising from the transverse curvature becomes of the same order of magnitude as the usual viscous stress term in the momentum equation. The downstream region is characterized by the fact that the effects produced by the transverse curvature can be considered to be essentially a perturbation of a flow which, in the limit of Δ/r_0 very much less than unity, approaches a two-dimensional pattern.

4. The downstream domain is divided into three sub-regions each one of which is a limiting case of the other. The first is where Δ/r_0 is very much less than unity and the effect of the axial-symmetry is negligible and the flow approaches a two-dimensional pattern. It is unlikely that this domain would be reached before

transition to turbulent flow took place. The second sub-region is where Δ/r_0 is small compared to unity, (say 0.1 or less), and this is where Mangler's formulation, which takes into account the transverse curvature effect only approximately, is valid. The third sub-region is where the boundary layer thickness begins to approach or become of the order of the body radius.

5. The Busemann integral of the two-dimensional energy equation for Prandtl number unity and an insulated surface, with an isoeenergetic free stream and arbitrary pressure gradient is valid under the same conditions for axially-symmetric flow, in which the transverse curvature effects are considered. The Crocco integral of the two-dimensional energy equation is also shown to hold for the constant surface temperature, zero pressure gradient case with Prandtl number unity.

6. By means of the coordinate transformation

$$\bar{x} = \int \frac{r_0'(x)}{L^2} dx \quad \text{and} \quad \bar{y} = \int \frac{r(x,y)}{L} dy$$

which generalizes Mangler's transformation, the boundary layer equations are reducible to an almost two-dimensional form. Here $r(x, y)$ is the distance from any point in the boundary layer to the axis of symmetry, and L is a characteristic reference length.

7. The additional term which arises in both the momentum and energy equations as a result of the transverse curvature, has the same effect as an external favorable pressure gradient, at least in

the region where Δ/r_0 is less than, or of the order of unity. On this basis results obtained by previous authors can be interpreted.

8. For a "near paraboloid", with the axial pressure gradient assumed to be zero, without making any approximations in the boundary layer equations "similar" profiles can be obtained. Here, similar is used in the restricted meaning that the velocity and temperature distributions are derivable from ordinary differential equations.

9. Within the region where Δ/r_0 is less than or of the order of unity, solutions of the axially-symmetric boundary layer equations for zero pressure gradient and body shapes where $r_0 = ax^n$ or $r_0 = ae^{bx}$ can be obtained as asymptotic series for the velocity and temperature in ascending powers of a parameter \tilde{F} . Here, \tilde{F} is small in comparison to unity, and is proportional to Δ/r_0 . The zeroth approximation is the Mangler result.

10. The first order correction to the Mangler formulation for $Pr = 1$ shows, that at least in the case of the cone and cylinder, the effect on both the skin friction coefficient and heat transfer rate can become appreciable in the range where Δ/r_0 is less than or of the order of unity. At a constant Δ/r_0 , the effects are increased in magnitude when either the ratio of wall to free stream temperature, or Mach number, is increased. Also, all other conditions being equal, for the same value of Δ/r_0 the skin friction coefficient (and hence heat transfer) increase on the cylinder is greater than

that on the cone. The displacement thickness for the cone and cylinder is decreased from the Mangler value, but only very slightly.

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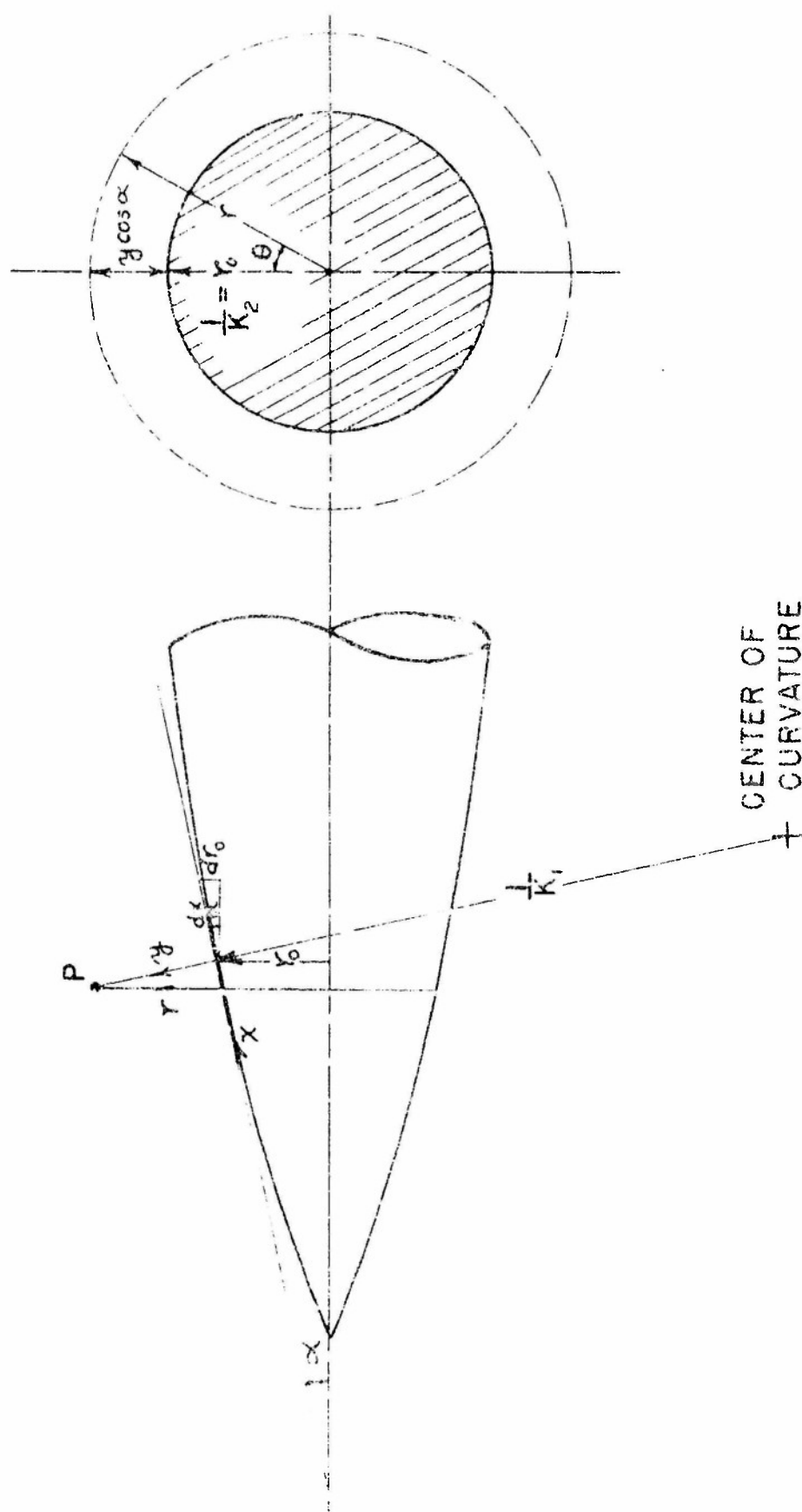
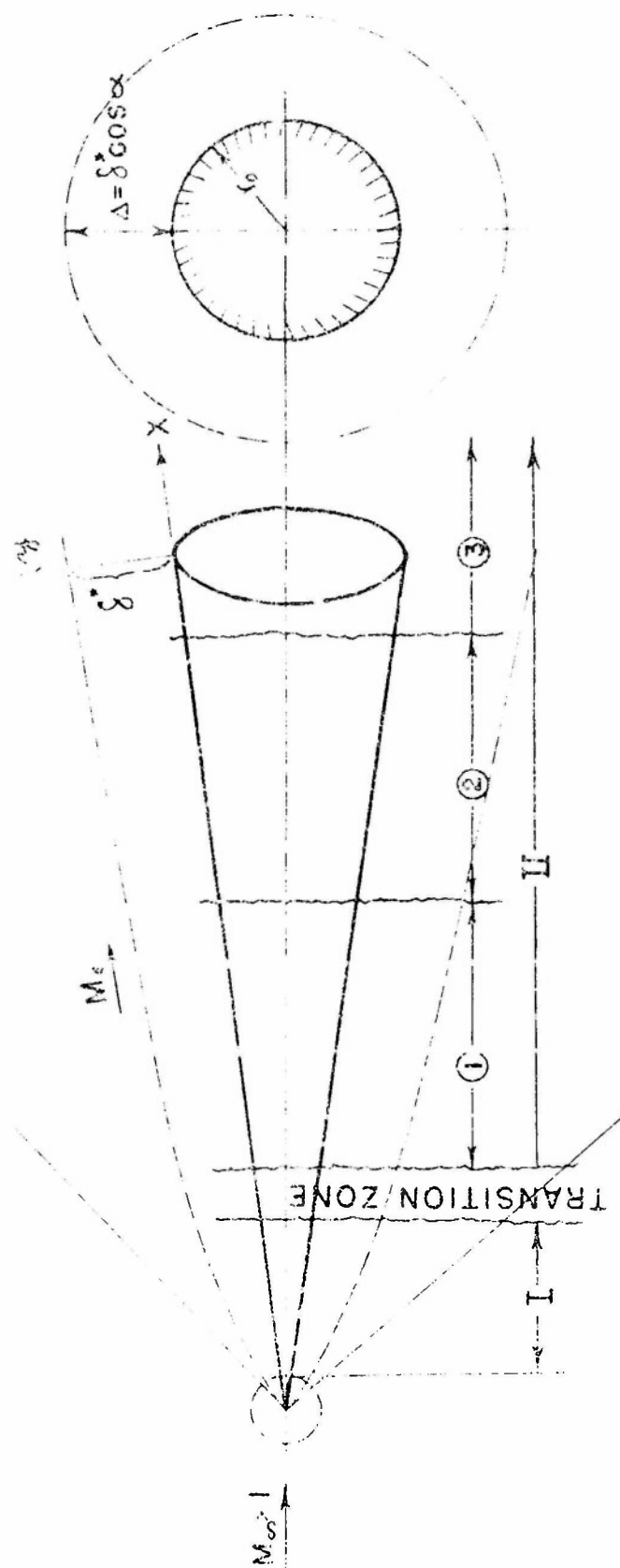


Figure 1 Coordinate System for Axially-Symmetric Body



○ IMMEDIATE NOSE REGION; $Re_\infty \sim 10 M^2$

I NOSE REGION; $\frac{\Delta}{r_0} \gg 1$

II DOWNSTREAM REGION

① $\frac{\Delta}{r_0} \gtrsim 1$

② $\frac{\Delta}{r_0} \ll 1$

③ $\frac{\Delta}{r_0} \ll \ll 1$

Figure 2 Main Flow Regions

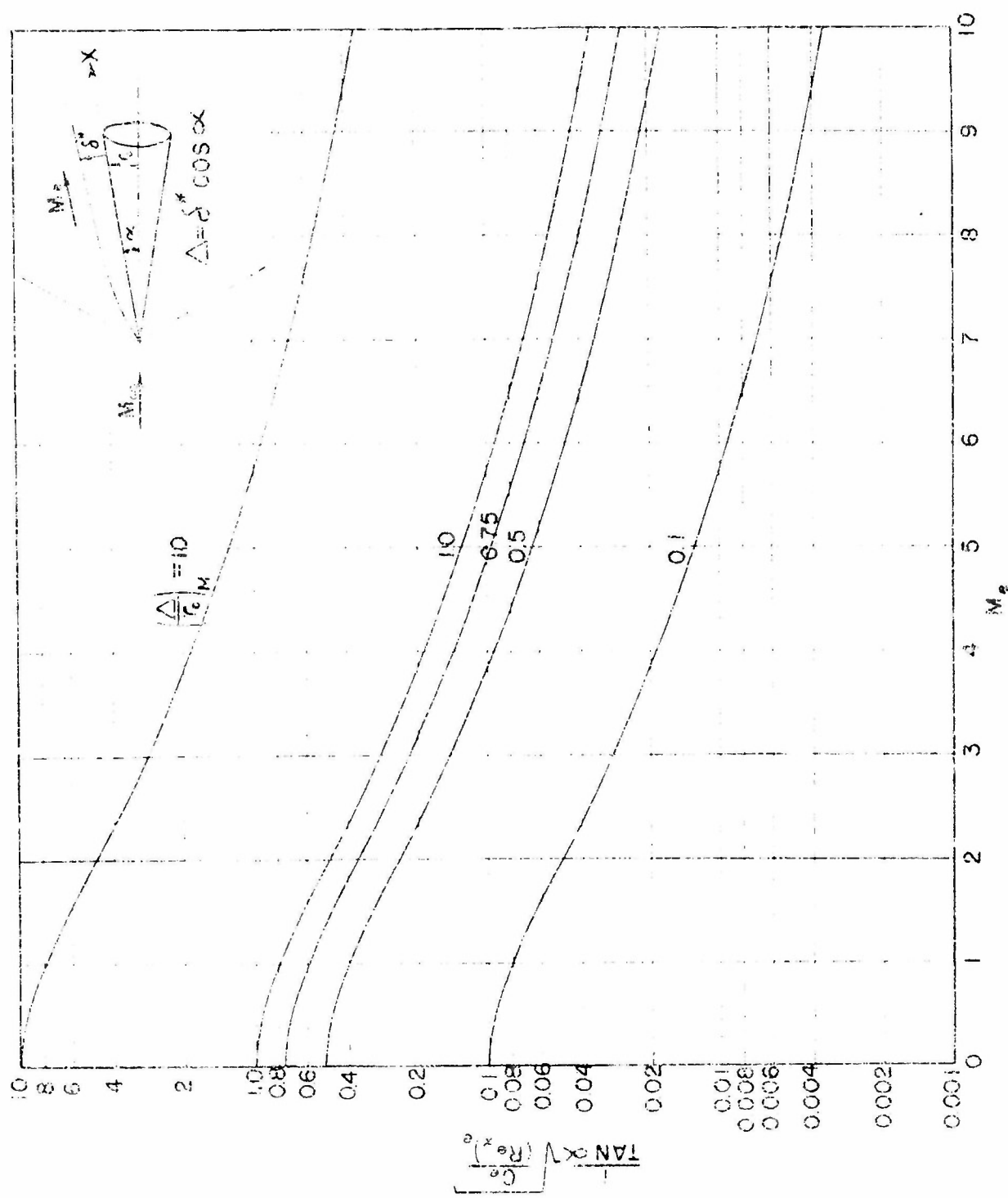


Figure 3 Displacement Thickness on an Insulated Cone for $Pr = 1$, $Z^* = 1.4$, Plotted as the Parameter Δ/r_o

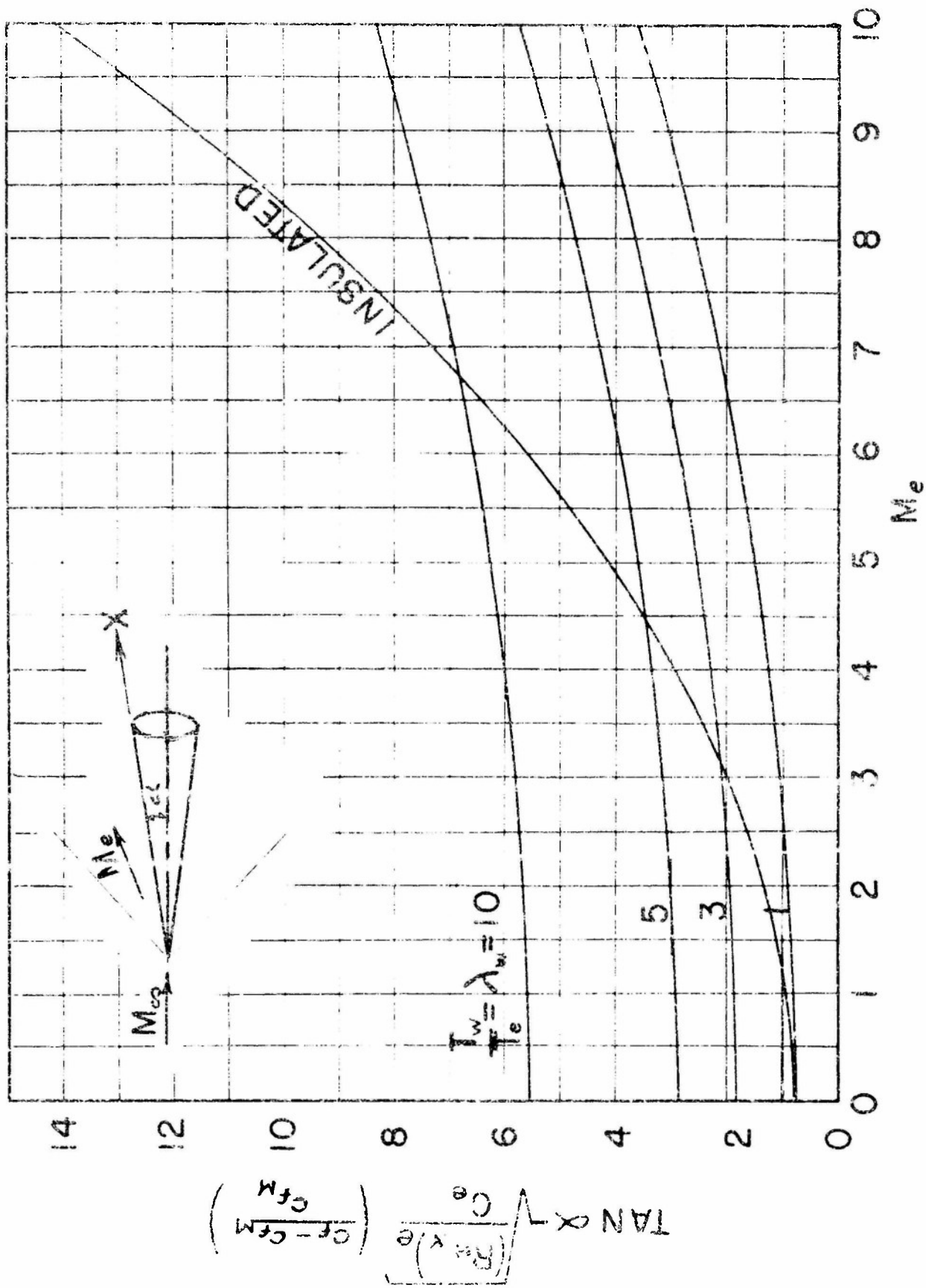


Figure 4 Skin Friction Coefficient on a Cone to Order Δ/r_0 for $Pr = 1$, $\gamma = 1.4$

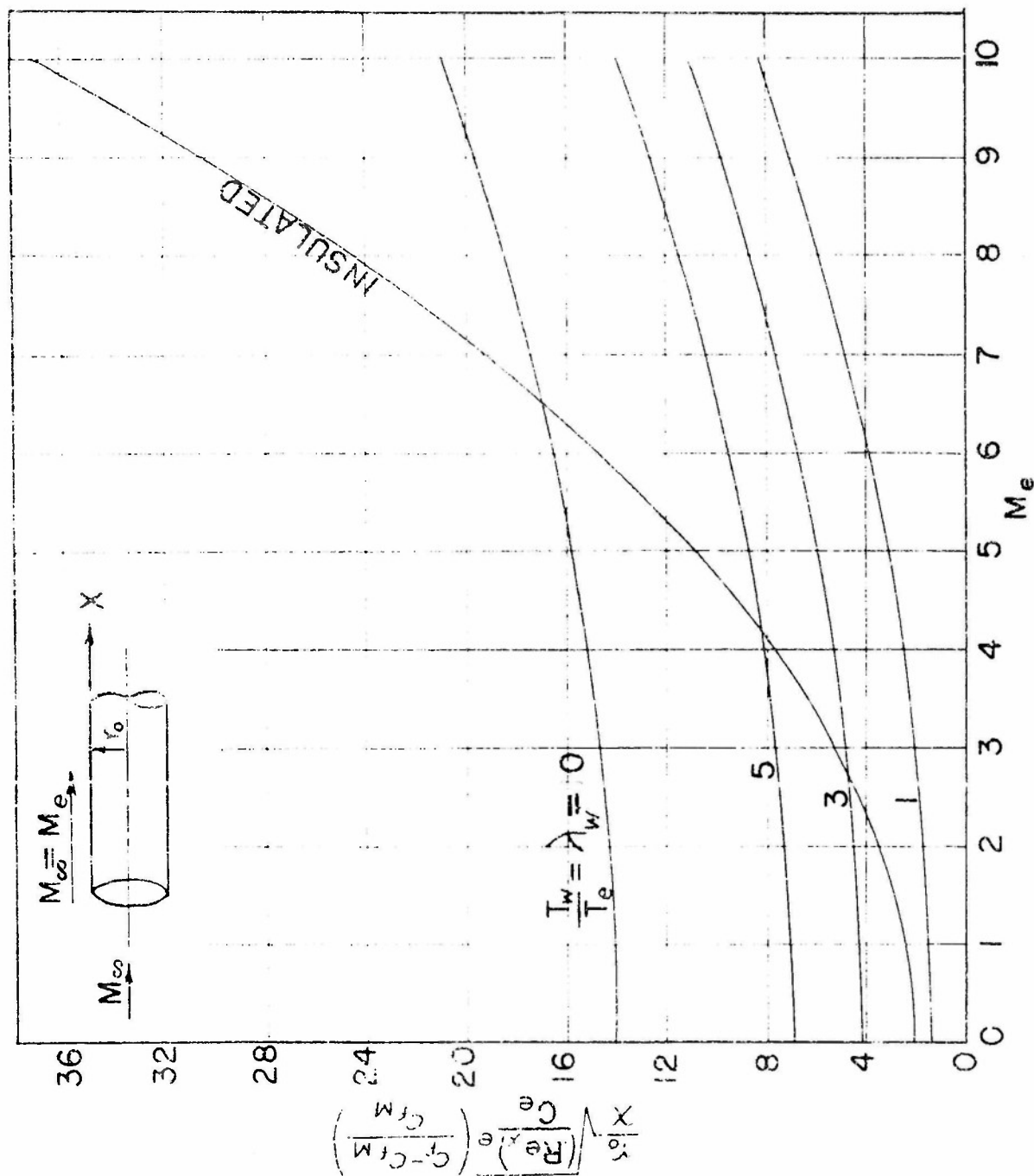


Figure 5 Skin Friction Coefficient on a Cylinder to Order Δ/r_0 for $Pr = 1$, $\gamma = 1.4$

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